Practical Linearization Control of Nonholonomic Unicycles

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Abstract

Due to underactuation, the states of nonholonomic systems cannot be steered toward any direction of the state space. This note takes unicycles as an example and demonstrates a new idea of control design for nonholonomic systems. More precisely, we apply state transformation technique and external dynamic oscillator, and convert an underactuated nonholonomic unicycle into a fully-actuated and linearizable one. A control law, capable of tracking and stabilization uses, is then constructed. The tradeoff therein is that the tracking/stabilization errors can only be steered into the neighborhood of the origin rather than converging to zero. Numerical simulations are carried out to validate the proposed control scheme.

Keywords: Practical linearization, Practical stabilization, Nonholonomic unicycle.

1. Introduction

The states of unicycles cannot be arbitrarily steered in the state space due to nonholonomic constraints caused by the lack of lateral control input [1]. The control laws developed for unicycles can be divided into two types: full-state (position + orientation) [2] [3] [4] [5] and position-only controllers [6] [7], both of which focus on either tracking or stabilization purposes.

Time-varying [2] or discontinuous technique [3] is generally applied to develop the stabilizers for unicycles. In [2], a time-related oscillating function is fused into the visual control law that achieves pose stabilization with the unknown focal length of the camera. Via combing the σ-process and switching mechanism, the discontinuous controller in [3] ensures exponential convergence of pose stabilization errors. For tracking uses in torque level, the backstepping control law proposed in [4] obtains uniformly asymptotical convergence of position and orientation errors. In [5], a simple switching control law with an invariant set capable of simultaneous tracking and stabilization is proposed. Additionally, the position-only control schemes in [6] [7] choose a point in front of the body center as a position coordinate and convert the position derivative into a feedback linearizable form, making linear control approaches applicable. However, the feedback linearization control laws like those in [6] [7] do not consider the zero dynamics associated with uncontrolled orientation. As a result, how orientation behaves in the steady state remains unknown. Therefore, it is significant to investigate the full-state feedback linearization schemes for unicycles.

Motivated by the discussion above, we generalize the state transformation technique used in our previous work [5] and convert the original kinematic model of the unicycle into a new feedback linearizable form by fusing pose states with external oscillators and introducing an additional control input. A simple proportional control law in terms of feedback linearization is then proposed. We prove that the pose error for either time-invariant or time-varying desired signals can be stabilized into a small ball enclosing the origin.

The remainder is organized as follows. Section 2 includes the problem formulation and control design. Numerical simulation results are presented in Section 3. Section 4 concludes the work briefly.

2. Main Results

2.1. Problem Formulation

The typical kinematic model of a unicycle can be described by [5],

$$\dot{x} = u\cos\theta, \dot{y} = u\sin\theta, \dot{\theta} = r \tag{1}$$

where $[x, y]^T$ is the position, θ denotes the orientation, u stands for the linear velocity, and r represents the angular velocity. Suppose that the desired pose signal $[x_d, y_d, \theta_d]^T$ obeys the same kinematics as (1) and satisfies the assumptions below,

Assumption 1. The time derivatives $\left[\dot{x}_d, \dot{y}_d, \dot{\theta}_d\right]^T \in L_\infty$.

Assumption 2. There exists a positive constant γ so that

$$\left\| [x_d, y_d, \theta_d] \right\|_2 \le \gamma \tag{2}$$

Then, the control objective can be stated as: under Assumptions 1-2, find a control law (u, r) such that

$$\lim \left\| [x, y, \theta] - [x_d, y_d, \theta_d] \right\| \le \delta \tag{3}$$

with a positive constant δ.

2.2. *Control Design*

First of all, we would like to convert the unicycle model (1) into a feedback linearizable form via state transformations. Define

$$\begin{cases} x_1 = x\cos\theta + y\sin\theta \\ y_1 = -x\sin\theta + y\cos\theta \\ \theta_1 = \theta \end{cases} \tag{4}$$

and calculate its derivative as below,

$$\dot{x}_1 = u + ry_1, \dot{y}_1 = -rx_1, \dot{\theta}_1 = r \tag{5}$$

For the purpose of practical linearization, an external dynamic oscillator in terms of auxiliary variables is needed. Let ϕ be an auxiliary variable and ε a small positive constant, we define the following new states

$$\begin{cases} z_1 = \theta_1 - \varepsilon\cos\phi \\ z_2 = x_1 - \varepsilon\sin\phi \\ z_3 = 2y_1 + \left(z_1 + \varepsilon\cos\phi\right)\left(z_2 + \varepsilon\sin\phi\right) \end{cases} \tag{6}$$

Calculating the time derivative of (6) leads to

$$\begin{cases} \dot{z}_1 = r + \varepsilon\dot{\phi}\sin\phi \\ \dot{z}_2 = u + ry_1 - \varepsilon\dot{\phi}\cos\phi \\ \dot{z}_3 = \varepsilon^2\dot{\phi} - \dot{z}_1\left(z_2 + 2\varepsilon\sin\phi\right) + \dot{z}_2\left(z_1 + 2\varepsilon\cos\phi\right) \end{cases} \tag{7}$$

Define new control inputs $w_1 \triangleq \dot{z}_1, w_2 \triangleq \dot{z}_2, w_3 \triangleq \varepsilon^2\dot{\phi}$, we convert (7) into

$$\left[\dot{z}_1, \dot{z}_2, \dot{z}_3\right]^T = A\left[w_1, w_2, w_3\right]^T \tag{8}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -z_2 - 2\varepsilon\sin\phi & z_1 + 2\varepsilon\cos\phi & 1 \end{bmatrix} \tag{9}$$

Obviously, the matrix A is invertible and the new model (8) is feedback linearizable. A more favorable feature is that the (8) is fully-actuated.

Following the (4) and (6), we perform some calculations on the desired signal and obtain,

$$\begin{cases} x_{1d} = x_d\cos\theta_d + y_d\sin\theta_d \\ y_{1d} = -x_d\sin\theta_d + y_d\cos\theta_d \\ \theta_{1d} = \theta_d \\ \dot{x}_{1d} = u_d + r_d y_{1d}, \dot{y}_{1d} = -r_d x_{1d}, \dot{\theta}_{1d} = r_d \end{cases} \tag{10}$$

Let

$$z_{1d} = \theta_{1d}, z_{2d} = x_{1d}, z_{3d} = 2y_{1d} + z_{1d}z_{2d} \tag{11}$$

and define errors,

$$e_1 = z_1 - z_{1d}, e_2 = z_2 - z_{2d}, e_3 = z_3 - z_{3d} \tag{12}$$

Before moving on, we propose the following lemma,

Lemma 1. There exists a positive constant δ so that (3) establishes, if $e_1 \to 0, e_2 \to 0, e_3 \to 0$ as $t \to +\infty$.

Proof. Using the conditions in the lemma and (6), one gets,

$$\begin{aligned} &\theta_1 - \theta_{1d} = \theta - \theta_d \to \varepsilon\cos\phi \\ &x_1 - x_{1d} \to \varepsilon\sin\phi \\ &y_1 - y_{1d} \to -\varepsilon\frac{z_{1d}\sin\phi + z_{2d}\cos\phi + \varepsilon\cos\phi\sin\phi}{2} \end{aligned} \tag{13}$$

For the term $y_1 - y_{1d}$, it is obvious that

$$\lim_{t\to\infty}\left\|y_1 - y_{1d}\right\| \le 0.5\varepsilon\left(\sqrt{z_{1d}^2 + z_{2d}^2} + \varepsilon\right) \tag{14}$$

which, together with the fact that $z_{2d}^2 \le x_d^2 + y_d^2$ and Assumption 2, leads to

$$\lim_{t\to\infty}\left\|y_1 - y_{1d}\right\| \le 0.5\varepsilon\left(\gamma + \varepsilon\right) \tag{15}$$

Therefore,

$$\left\|\left[x_1 - x_{1d}, y_1 - y_{1d}\right]\right\| \le 0.5\varepsilon\left(\gamma + \varepsilon + 2\right) \tag{16}$$

Let $S(*) = \begin{bmatrix} \cos* & \sin* \\ -\sin* & \cos* \end{bmatrix}$, we know that

$$\begin{aligned} \begin{bmatrix} x_1 - x_{1d} \\ y_1 - y_{1d} \end{bmatrix} &= S(\theta)\begin{bmatrix} x \\ y \end{bmatrix} - S(\theta_d)\begin{bmatrix} x_d \\ y_d \end{bmatrix} \\ &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x - x_d \\ y - y_d \end{bmatrix} \\ &\quad + \left[S(\theta) - S(\theta_d)\right]\begin{bmatrix} x_d \\ y_d \end{bmatrix} \end{aligned} \tag{17}$$

and

$$\begin{aligned} S(\theta) - S(\theta_d) &= \begin{bmatrix} \cos\theta - \cos\theta_d & \sin\theta - \sin\theta_d \\ -\sin\theta + \sin\theta_d & \cos\theta - \cos\theta_d \end{bmatrix} \\ &= 2\sin\frac{\theta - \theta_d}{2}\begin{bmatrix} -\sin\frac{\theta + \theta_d}{2} & \cos\frac{\theta + \theta_d}{2} \\ -\cos\frac{\theta + \theta_d}{2} & -\sin\frac{\theta + \theta_d}{2} \end{bmatrix} \end{aligned}$$

where the sum-to-product formula is used. Henceforth,

$$\left\|\left[S(\theta) - S(\theta_d)\right]\begin{bmatrix} x_d \\ y_d \end{bmatrix}\right\| \le \left\|\theta - \theta_d\right\|\left\|\left[x_d, y_d\right]\right\| \tag{18}$$

By Assumption 2 and the fact $\left\|\left[x_d, y_d\right]\right\| \le \left\|\left[x_d, y_d, \theta_d\right]\right\|$, we use (17) and obtain

$$\lim_{t\to+\infty}\left\|\left[x - x_d, y - y_d\right]\right\| \le 0.5\varepsilon\left(3\gamma + \varepsilon + 2\right) \tag{19}$$

The ultimate bound δ can then be estimated by,

$$\delta = 0.5\varepsilon\left(3\gamma + 3\varepsilon + 2\right) \tag{20}$$

The claims in the lemma hence establish. □

Let $K = \operatorname{diag}\left\{k_1, k_2, k_3\right\}$ be a positive definite matrix, we design the control inputs as,

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = A^{-1}\left\{\begin{bmatrix} \dot{z}_{1d} \\ \dot{z}_{2d} \\ \dot{z}_{3d} \end{bmatrix} - K\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}\right\} \tag{21}$$

The original control inputs can be recovered via the following procedures,

$$\dot{\phi} = \frac{w_3}{\varepsilon^2}, r = w_1 - \varepsilon\dot{\phi}\sin\phi, u = w_2 - r y_1 + \varepsilon\dot{\phi}\cos\phi \qquad (22)$$

Then, we summarize the main results of the current work in the theorem below,

Theorem 1. Given Assumptions 1-2, the application of the control law (21)(22) on the unicycle (1) achieves the control objective (3).

Proof. Using (22) into (21), we obtain,

$$\dot{e}_1 = -k_1 e_1, \dot{e}_2 = -k_2 e_2, \dot{e}_3 = -k_3 e_3 \qquad (23)$$

which demonstrates that $[e_1, e_2, e_3]^T$ converges to zero exponentially [8]. Using Lemma 1, we conclude that the control objective (3) is achieved. ☐

Remark 1. The pose errors in steady state are not stabilized to zero and the actual pose states would converge into the neighborhood around the desired signal, see Fig.1.

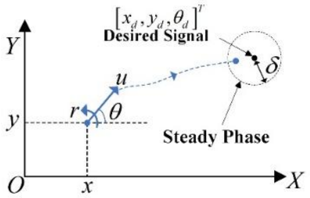


Fig.1 Pose trajectories in steady state.

3. Numerical Simulation

In this section, we perform the simulation with three different desired signals to verify the control scheme. The control coefficients for all cases are selected as $\varepsilon = 0.2, k_1 = k_2 = k_3 = 5.5$ identically. We initialize the unicycle states by $x(0) = -4, y(0) = -4.5, \theta(0) = -0.5\pi$. We choose the following desired signals:

Case 1. Circular trajectory

$$\dot{x}_d = 0.2\cos\theta_d, \dot{y}_d = 0.2\sin\theta_d, \dot{\theta}_d = 0.025$$

$$x_d(0) = 0, y_d(0) = -8, \theta_d(0) = 0$$

Case 2. Straight line

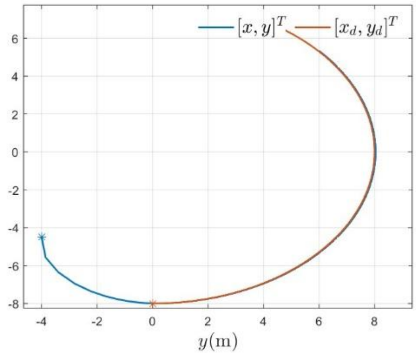
$$\dot{x}_d = 0.2\cos\theta_d, \dot{y}_d = 0.2\sin\theta_d, \dot{\theta}_d = 0$$

$$x_d(0) = -3.5, y_d(0) = -3, \theta_d(0) = 0.25\pi$$

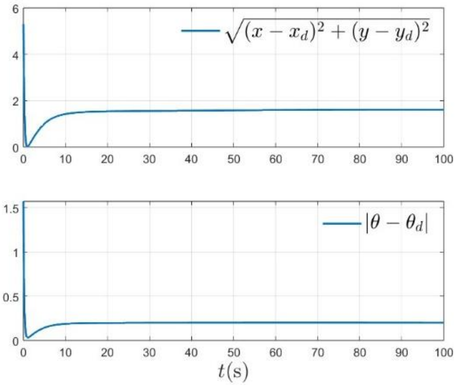
Case 3. Fixed point

$$x_d \equiv 0, y_d \equiv 0, \theta_d \equiv 0$$

The results are depicted in Fig. 2-4. As can be seen, all results accord with the theoretical analysis and the unicycle can stabilize itself to the neighborhood of the desired signal that can be either time-varying or time-invariant. The position trajectories seem to be zig-zag during transient phase, due to the auxiliary variable ϕ is oscillating for compensating the underactuation. Note also that the ultimate bounds of pose tracking errors can be reduced by decreasing ε, which can be drawn from Lemma 1 and (20).

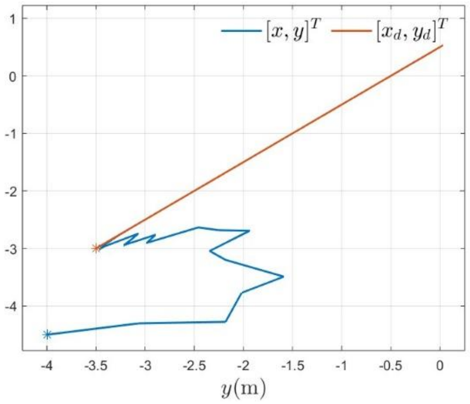


a .Position trajectories (*: starting point)

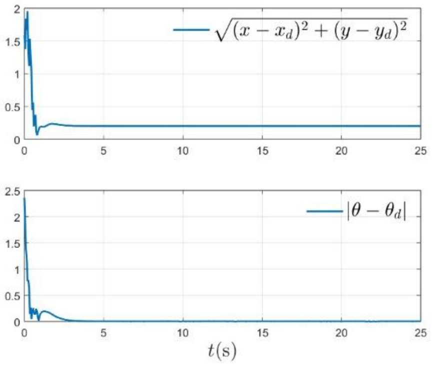


b . Pose error trajectories

Fig.2 Simulation results of Case 1.

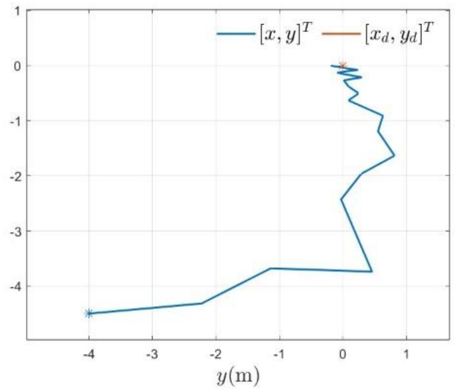


a . Position trajectories (*: starting point)

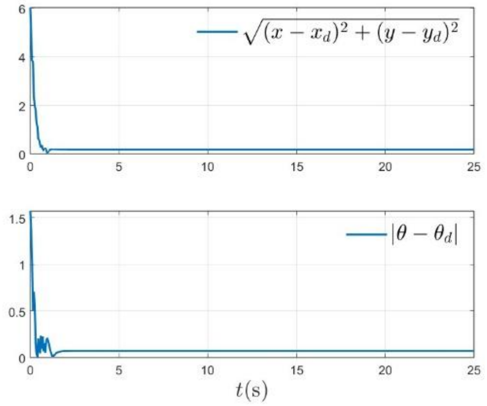


b . Pose error trajectories

Fig.3 Simulation results of Case 2.



a . Position trajectories (*: starting point)



b . Pose error trajectories

Fig.4 Simulation results of Case 3.

4. Conclusion

This work proposes a practical linearization control law for nonholonomic unicycles. The underactuated property is compensated by an additional control input fused with external oscillators after state transformations. The feedback linearizable form of the unicycle system is also illustrated. It of worthy noting that the errors with respect to the desired pose signal can converge to a small ball enclosing the origin. In the future, we will generalize the presented control law to the coordinated control of networked unicycles.

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References

1. L. Yan and B. Ma, Adaptive practical leader-following formation control of multiple nonholonomic wheeled mobile robots, International Journal of Robust Nonlinear Control, Vol.30(17), 2020, pp.7216-7237.
2. X. Zhang, Y. Fang, and N. Sun, Visual servoing of mobile robots for posture stabilization: from theory to experiments, International Journal of Robust and Nonlinear Control, Vol. 25(1), pp.1-15, 2015.
3. A. Astolfi, Discontinuous control of nonholonomic systems, Systems and Control Letters, Vol.27 (1), pp.37-45, 1996.
4. R. Fierro and F. L. Lewis, Control of a nonholonomic mobile robot: Backstepping kinematics into dynamics, Journal of Robotic Systems, Vol.14(3), pp.149-163, 1998.
5. L. Yan and B. Ma, Universal Control for Both Rendezvous and Tracking of Multiple Nonholonomic Unicycles, IEEE Transactions on Control of Network Systems, Vol.11(1), 2024, pp. 439-449.
6. W. Xie and B. Ma, Position centroid rendezvous and centroid formation of multiple unicycle agents, IET Control Theory and Applications, Vol.8 (17), pp.2055–2061, 2014.
7. M. A. Maghenem, A. Loria, and E. Panteley, Cascades-based leaderfollower formation tracking and stabilization of multiple nonholonomic vehicles, IEEE Transactions on Automatic Control, Vol.65 (8), pp. 3639–3646, 2019.
8. H.K., Khalil, Nonlinear Systems (3rd Edition), Patience Hall, 2002.

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