Global Stabilization of A Class of Nonholonomic Integrators via Discontinuous Control

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Abstract

This paper investigates the discontinuous state feedback control for stabilizing a class of nonholonomic integrators with drift terms. The control design relies on constraining state trajectory in an invariant set. To this end, we apply constant controls to drive the states moving into the invariant set and then switch to a continuous control law with suitable gain selections. It is proven in the Lyapunov sense that the proposed control scheme achieves global exponential stabilization of the states, and the control switch would only occur at most once. Numerical simulations are carried out to validate the proposed control law.

Keywords: Nonholonomic integrators, Discontinuous feedback, Nonlinear control

1. Introduction

Nonholonomic integrators refer to one kind of control systems that obstruct Brockett's necessary condition for the existence of static time-invariant stabilizers [1]. They have attracted much attention in the control community because various robotic systems can be converted into nonholonomic integrators [2].

Due to non-integrable properties, only discontinuous, time-varying, or hybrid control laws are applicable to stabilize the nonholonomic integrators [3], [4], [5], [6]. Via forcing the state trajectories to move on a sliding mode surface, a stabilization and a tracking control law were developed in [3] for nonholonomic integrators, achieving global asymptotical convergence of the states. In [4], a novel logic-based hybrid control law with the switching mechanism that achieved global exponential stabilization was reported. Using virtual control and variable structure design, the control scheme in [5] stabilized the states of nonholonomic integrators to zero exponentially from any initial states. An alternative with the control Lyapunov function approach for stabilizing the nonholonomic integrators can be found in [6]. The leading results in [3], [4], [5], [6] are helpful in understanding the structural properties and solution trajectory of nonholonomic integrators. Yet, the drift terms that play affect on the dynamic performance of the system are not considered in the literature mentioned above.

It is necessary to consider drift terms for utilizing the results developed for pure nonholonomic integrators on practical robotic systems. In [2], an adaptive leader-following formation control scheme of multiple wheeled mobile robots was developed by applying techniques associated with nonholonomic integrators, regulating formation errors globally convergent to the neighborhood of the origin. After converting the kinematic model of unicycles into the form of nonholonomic integrators with drift terms, a control law capable of rendezvous and tracking of networked unicycles was proposed in [7], which, however, only consider the case that the error states are initialized inside an invariant set.

Motivated by the discussions above, this paper makes further endeavors on the control design of nonholonomic integrators. The concerned nonholonomic integrators can be viewed as an augmented version of that in [1] by adding a drift term. The control design involves a state feedback control law and a constant control law, which achieves asymptotical convergence of the states in an invariant set and forces the state trajectory to move into the invariant set, respectively. A simple switch would occur if the initial states are outside the invariant set. Lyapunov stability theory is utilized to prove the obtained theoretical results.

The rest is organized as follows. Section 2 formulates the control problem and presents the control design.

Numerical simulations are carried out in Section 3. Section 4 concludes the work briefly.

2. Main results

2.1. Problem formulation

The concerned nonholonomic integrator with a drift term in this work holds the form below,

$$\begin{cases} \dot{z}_1 = u_1 \\ \dot{z}_2 = u_2 \\ \dot{z}_3 = z_2 u_1 - z_1 u_2 + f(z_1, z_2) \end{cases} \tag{1}$$

where $z_1, z_2, z_3, u_1, u_2 \in \mathbb{R}$, $f : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$, and there is a positive number κ_1 such that $|f(z_1, z_2)| \le \kappa_1 \|[z_1, z_2]\|$. The nonholonomic dynamic (1) obstructs the famous Brockett's necessary condition for the existence of full-state time-invariant static stabilizer. To this end, we plan to find a discontinuous control law for (1) so that

$$\lim_{t\to\infty} z_1 = 0, \lim_{t\to\infty} z_2 = 0, \lim_{t\to\infty} z_3 = 0 \tag{2}$$

from any initial states.

Remark 1. *Either kinematic or dynamic models of various nonholonomic systems, including nonholonomic unicycles and underactuated hovercrafts [7], can be converted into the form of (1). Thus, the addressed nonholonomic integrator (1) has general property though its form is simple.*

2.2. *Control design*

The discontinuous control design includes two steps. First, we design a state feedback control law so that the state trajectory is convergent to zero in an invariant set. Second, we propose a constant control to force the states to move into the invariant set. The design process above is depicted in Fig. 1.

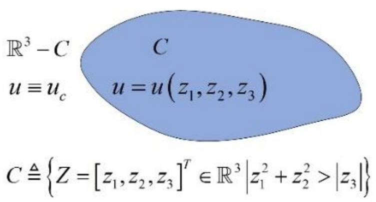

Fig. 1. The invariant set and switching mechanism.

Design the following control law,

$$\begin{aligned} u_1 &= -k_1 \tanh z_1 - \frac{z_2}{z_1^2 + z_2^2}(f + g + k_2 z_3) \\ u_2 &= -k_1 \tanh z_2 + \frac{z_1}{z_1^2 + z_2^2}(f + g + k_2 z_3) \end{aligned} \tag{3}$$

where $g \triangleq -k_1 z_2 \tanh z_1 + k_1 z_1 \tanh z_2, k_2 > 2k_1 > 0$. The effectiveness of the control law (3) is summarized in the lemma below.

LEMMA 1. *The application of (3) on (1) achieves that $z_1 \to 0, z_2 \to 0, z_3 \to 0$ as $t \to +\infty$ and $u_1, u_2 \in L_\infty$, if $k_2 > 2k_1 > 0$ and $Z(0) \in C \triangleq \{ Z \in \mathbb{R}^3 \,|\, z_1^2 + z_2^2 > |z_3| \}$.*

Proof. Substituting (3) into (1) results in,

$$\begin{aligned} \dot{z}_1 &= -k_1 \tanh z_1 - \frac{z_2}{z_1^2 + z_2^2}(f + g + k_2 z_3) \\ \dot{z}_2 &= -k_1 \tanh z_2 + \frac{z_2}{z_1^2 + z_2^2}(f + g + k_2 z_3) \\ \dot{z}_3 &= -k_2 z_3 \end{aligned} \tag{4}$$

Obviously, one has $z_3(t) = z_3(0)e^{-k_2 t}$ and obtains that z_3 would converge to zero exponentially. Choose a positive function,

$$V = 0.5(z_1^2 + z_2^2) \tag{5}$$

The time derivative of V along with the solution trajectory of (4) can be calculated as,

$$\dot{V} = -k_1 z_1 \tanh z_1 - k_1 z_2 \tanh z_2 \le 0 \tag{6}$$

which implies that the states z_1 and z_2 converge to zero asymptotically. Next, we prove the boundedness of the control inputs and invariant properties of the set C. Due to the forms of f and g, it is direct to obtain

$$\begin{aligned} |f| &\le \kappa_1 \sqrt{z_2^2 + z_2^2} \\ |g| &\le k_1 |z_1| + k_1 |z_2| \le 2k_1 \sqrt{z_2^2 + z_2^2} \end{aligned} \tag{7}$$

which, together with (3), demonstrates that

$$\begin{aligned} |u_1| &\le k_1 + \frac{|f| + |g|}{\sqrt{z_1^2 + z_2^2}} + \frac{k_2 |z_3|}{\sqrt{z_1^2 + z_2^2}} \\ &\le 3k_1 + \kappa_1 + \frac{k_2 |z_3|}{\sqrt{z_1^2 + z_2^2}} \end{aligned} \tag{8}$$

According to $|x| \ge |\tanh x|, \forall x \in \mathbb{R}$, we derive that

$$\dot{V} \ge -k_1(z_1^2 + z_2^2) = -2k_1 V \tag{9}$$

Using the comparison principle then yields [8],

$$V(t) \ge e^{-2k_1 t} V(0) \tag{10}$$

Therefore, one has

$$\|[z_1(t), z_2(t)]\| \ge \|[z_1(0), z_2(0)]\| e^{-k_1 t} \tag{11}$$

and

$$\frac{|z_3|}{\sqrt{z_1^2 + z_2^2}} \le \frac{|z_3(0)|}{\sqrt{z_1^2(0) + z_2^2(0)}} \tag{12}$$

Combining (8) and (12), we find out that

$$|u_1| \le U \triangleq 3k_1 + \kappa_1 + k_2 \frac{|z_3(0)|}{\sqrt{z_1^2(0) + z_2^2(0)}} \tag{13}$$

and u_1 is bounded. Following the same routine above, we can also prove the boundedness $|u_2| \le U$. Additionally, the inequality (11) and the fact $z_3(t) = z_3(0)e^{-k_2 t}$ imply that $z_1^2 + z_2^2 > |z_3|, \forall t \ge 0$ as $k_2 > 2k_1$ and $Z(0) \in C$.

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Therefore, the set C is invariant. This completes the proof. □

The Lemma 1 shows that the state trajectory would converge to zero asymptotically in the invariant set C. In what follows, we would like to use constant controls to drive the state trajectory moving into the set C if the states are not initialized therein.

Without losing generality, the initial states are assumed to be outside the invariant set C, i.e., $Z(0) \in \mathbb{R}^3 - C$. Consider the constant control inputs,

$$u_1 \equiv u_{1c}, u_2 \equiv u_{2c} \tag{14}$$

Using (14) and integrating (1) with respect to time then lead to,

$$\begin{aligned} z_1(t) &= z_1(0) + u_{1c}t \\ z_2(t) &= z_2(0) + u_{2c}t \\ z_3(t) &= z_3(0) + \left[z_2(0)u_{1c} - z_1(0)u_{2c} \right]t \\ &\quad + \int_0^t f\left(z_1(\tau), z_2(\tau)\right)\mathrm{d}\tau \end{aligned} \tag{15}$$

The term $\int_0^t f\left(z_1(\tau), z_2(\tau)\right)\mathrm{d}\tau$ satisfies,

$$\begin{aligned} \int_0^t f\left(z_1(\tau), z_2(\tau)\right)\mathrm{d}\tau &\le \kappa_1 \int_0^t \left(|z_1(0)| + |z_2(0)|\right)\mathrm{d}\tau \\ &\quad + \kappa_1 \int_0^t \left(|u_{1c}| + |u_{2c}|\right)\tau \mathrm{d}\tau \\ &= c_1 t + c_2 t^2 \end{aligned} \tag{16}$$

where $c_1 = \kappa_1\left(|z_1(0)| + |z_2(0)|\right), c_2 = 0.5\kappa_1\left(|u_{1c}| + |u_{2c}|\right)$.

Let $c_0 = |z_3(0)|, c_3 = \kappa_1\left(|z_1(0)| + |z_2(0)|\right) + \left[z_2(0)u_{1c} - z_1(0)u_{2c}\right]$, the estimation for $z_3(t)$ can be given by,

$$z_3(t) \le c_0 + c_3 t + c_2 t^2 \tag{17}$$

It hence follows that,

$$\begin{aligned} z_1^2(t) + z_2^2(t) - |z_3(t)| &\ge \left(z_1(0) + u_{1c}t\right)^2 + \left(z_2(0) + u_{2c}t\right)^2 \\ &\quad - c_0 - c_3 t - c_2 t^2 \\ &= b_2 t^2 + b_1 t + b_0 \end{aligned} \tag{18}$$

with

$$\begin{aligned} b_2 &\triangleq u_{1c}^2 + u_{2c}^2 - c_2, \\ b_1 &\triangleq 2z_1(0)u_{1c} + 2z_2(0)u_{2c} - c_3 \\ b_0 &\triangleq z_1^2(0) + z_2^2(0) - c_0 < 0 \end{aligned} \tag{19}$$

Observing (19), if u_{1c}, u_{2c} are chosen satisfying $u_{1c}^2 + u_{2c}^2 > c_2$, then we conclude from $b_2 > 0$ and $b_0 < 0$ that there exists a finite time instant $t_1 > 0$ so that,

$$\begin{aligned} z_1^2(t_1) + z_2^2(t_1) - |z_3(t_1)| &= 0 \\ z_1^2(t) + z_2^2(t) - |z_3(t)| &> 0, \forall t > t_1 \end{aligned} \tag{20}$$

Such a t_1 can be estimated as,

$$0 < t_1 \le t_2 \triangleq \frac{-b_1 + \sqrt{b_1^2 - 4b_2 b_0}}{2b_2} \tag{21}$$

The derivations above are gathered together in the following lemma.

LEMMA 2. *Given the constant control inputs $[u_{1c}, u_{2c}]$ in (14) satisfying $u_{1c}^2 + u_{2c}^2 > c_2$ and $Z(0) \in \mathbb{R}^3 - C$, there is a finite time instant t_1 given by (21) so that $Z(t_1) \in C$.*

Proof. The proof is direct via following the derivations (14)-(21), and hence omitted. □

According to Lemmas 1-2, the discontinuous control law can be constructed as

$$\begin{aligned} u_1 &= \begin{cases} -k_1 \tanh z_1 - \dfrac{z_2}{z_1^2 + z_2^2}\left(f + g + k_2 z_3\right), \text{if } Z \in C \\ u_{1c}, \qquad\qquad \text{if } Z \in \mathbb{R}^3 - C \end{cases} \\ u_2 &= \begin{cases} -k_1 \tanh z_2 + \dfrac{z_1}{z_1^2 + z_2^2}\left(f + g + k_2 z_3\right), \text{if } Z \in C \\ u_{2c}, \qquad\qquad \text{if } Z \in \mathbb{R}^3 - C \end{cases} \end{aligned} \tag{22}$$

THEOREM 1. *Given the discontinuous control law (22), the system (1) is globally asymptotically stable if $k_2 > 2k_1 > 0$ and $u_{1c}^2 + u_{2c}^2 > c_2$.*

Proof. According to Lemma 1, the states would converge to zero asymptotically in the invariant set C by the control law (22). Moreover, the states would be driven moving into the set C in finite time if they are initialized outside the invariant set. Thus, the discontinuous scheme (22) would always ensure the asymptotic convergence to zero of the states. The claims in the theorem are established immediately. □

Remark 2. *Note that the time instant t_1 is not necessarily known for control switch in (22). We introduce the estimation on t_1 to prove that the state trajectory would move into the invariant set C in finite time.*

Remark 3. *Due to the invariant property of C, the switch of the control law (22) would only occur once at most.*

2.3. *Application to the tracking control of a unicycle robot*

This subsection illustrates the application of (22) on the trajectory tracking of a unicycle robot. The kinematic model of a unicycle robot can be given by

$$\dot{x} = v\cos\theta, \dot{y} = v\sin\theta, \dot{\theta} = \omega \tag{23}$$

where $[x, y]^T$ denotes the Cartesian position, θ is the orientation angle, u represents linear velocity and ω stands for the angular velocity. Let $[x_r, y_r, \theta_r]^T$ denote the reference trajectory generated by

$$\dot{x}_r = v_r \cos\theta_r, \dot{y}_r = v_r \sin\theta_r, \dot{\theta}_r = \omega_r \tag{24}$$

where the reference velocities v_r and ω_r are bounded. Define the following tracking errors,

$$\begin{aligned} z_1 &= (x - x_r)\cos\theta + (y - y_r)\sin\theta \\ z_2 &= \theta - \theta_r \\ z_3 &= -2(x - x_r)\sin\theta + 2(y - y_r)\cos\theta + z_1 z_2 \end{aligned} \tag{25}$$

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The time derivative of (25) can be calculated as

$$\begin{aligned} \dot{z}_1 &= v + 0.5\omega(z_3 - z_1 z_2) - v_r \cos z_2 \\ \dot{z}_2 &= \omega - \omega_r \\ \dot{z}_3 &= z_2 \dot{z}_1 - z_1 \dot{z}_2 + 2(v_r \sin z_2 - \omega_r z_1) \end{aligned} \quad (26)$$

Obviously, the error dynamics (26) features the same structure as (1) if we define

$$\begin{aligned} u_1 &= v + 0.5\omega(z_3 - z_1 z_2) - v_r \cos z_2 \\ u_2 &= \omega - \omega_r \\ f &= 2(v_r \sin z_2 - \omega_r z_1) \\ \kappa_1 &= 2\sqrt{v_r^2 + \omega_r^2} \end{aligned} \quad (27)$$

Therefore, applying the control law (22) on (26) would steer the error (25) to zero globally asymptotically. As the state transformation (25) is globally invertible, the original tracking errors $x - x_r, y - y_r$ and $\theta - \theta_r$ would converge to zero asymptotically from any initial states.

3. Numerical Simulations

This section validates the proposed control law (22) by numerical simulations. To this end, we initialize the nonholonomic integrator (1) by $Z(0) = [-1, 1, -4]^T$ and set the control coefficients as $k_1 = 0.25, k_2 = 0.55$. The drift term is $f = 0.2(z_1 + z_2)$. The constant control inputs are $u_{1c} = 0.2, u_{2c} = 0.2$. We depict the results in Figs 2-3.

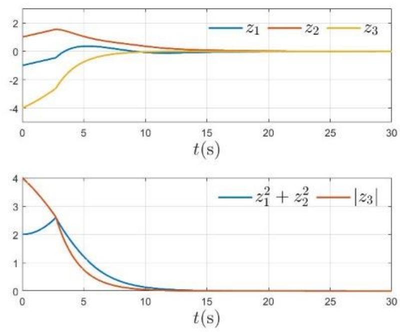

Fig. 2. The state trajectories.

As can be seen from Fig. 2. the initial states are outside the invariant set, and the control inputs are constant in this phase. The control is automatically switched into state feedback control law as soon as the switching condition $z_1^2(t) + z_2^2(t) > |z_3(t)|$ is satisfied, which can be drawn from the turnings in Fig. 2. Then, the states are constrained in the invariant set and converge to zero asymptotically. In addition, the control inputs are always bounded, which can be concluded from Fig. 3.

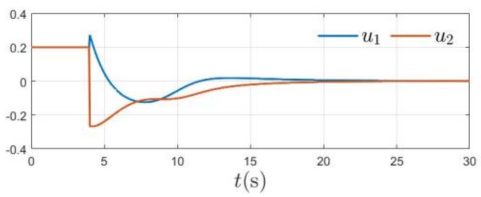

Fig. 3. The control inputs.

To validate the control law on solving the trajectory tracking control problem of a unicycle robot, we set the reference signal by $v_r = 0.2, \omega_r = 0.1$ with initial condition $x_r(0) = 0, y_r(0) = -2, \theta_r(0) = 0$. The initial pose of the unicycle is chosen as $x(0) = -2, y(0) = -2.5, \theta(0) = \pi$. Meanwhile, the control coefficients k_1, k_2, u_{1c} and u_{2c} are set the same as previous case. We depict the position trajectory and errors in Fig. 4 and Fig. 5, respectively.

As can be seen, the Fig. 4-5 illustrate the success of applying discontinuous control law (22) on the trajectory tracking control of a typical unicycle robot. The pose tracking errors are convergent to zero asymptotically.

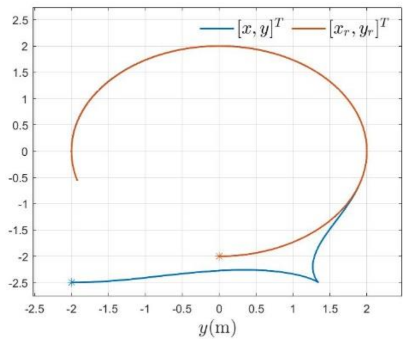

Fig. 4. The position trajectory(*:starting point).

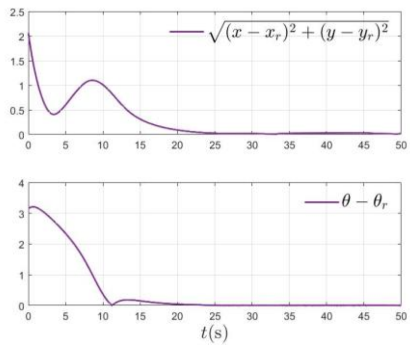

Fig. 5. The tracking errors.

4. Conclusion

This brief introduces a discontinuous global stabilizer for a class of nonholonomic integrators with drift terms. The proposed control scheme includes a state feedback controller and a constant control law, fusing with a simple criterion for control switching. It is proven in the Lyapunov sense that the system states converge to zero globally asymptotically, with the control switching occurring at most once. In the future, the authors will generalize the current control scheme to solve global stabilization problems of other nonholonomic systems.

Acknowledgements

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