A High-Speed Estimation Method of Parameters in Impulse Response

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Abstract

This paper proposes a high-speed parameters estimation method for compartment model where output function is described by the convolution between input function and impulse response, which is like a time-invariant linear system. The proposed method uses linear regression analysis based on the equivalently transformed equation that can be obtained using the processing of Differentiation of Convolution with Exponential function (DCE). In this paper, taking the parameters estimation problem of PET (Positron Emission Tomography) inspection system and RLC series electrical circuit for examples, we show that the method can estimate parameters of those impulse responses in high speed.

Keywords: Parameter estimation, Compartment model, Cumulative function, Linear regression analysis, PET Inspection

1. Introduction

Compartment model has been used in the various fields such as pharmacokinetics, chemical reaction system, environmental diffusion, and electrical circuit, etc. Especially, the estimation of parameters in PET (Positron Emission Tomography) inspection is very important in practice ([1], [2], [3], [4], [5], [6], [7], [8]).

In this paper, we propose a high-speed parameters estimation method for compartment model where output function is described by the convolution between input function and impulse response, which is like a time-invariant linear system ([9], [10], [11], [12]). The proposed method uses linear regression analysis based on an equivalent equation that can be obtained from DCE (Differentiation of Convolution with Exponential function).

Moreover, taking the parameters estimation problem of PET inspection system and RLC series electrical circuit for examples, we concretely describe that the method can estimate parameters of those impulse responses in high-speed.

2. Proposed Estimation Method

2.1 Impulse Response of PET Compartment Model

The compartment model and parameters used in the PET inspection is shown in Fig. 1. And the simultaneous differential equations of the 3-comparement model is shown in Eq. (1). The $C_p(t)$ means the tracer's radioactivity concentration in plasma that can be directly observed by arterial blood sampling from the patient after intravenous injection of FDG (^{18}F-fluorodeoxyglucose) tracer. The $C_e(t)$ means the tracer concentration before metabolism in tissue, and $C_m(t)$ is the tracer concentration after metabolism. $C_i(t)$ is measured by PET camera as the sum of $C_e(t)$ and $C_m(t)$. The flow parameters K_1, k_2, k_3, k_4 ([$ml \cdot g^{-1} \cdot min^{-1}$] or [$min^{-1}$]) show the transport and binding rates of the tracer between two compartments.

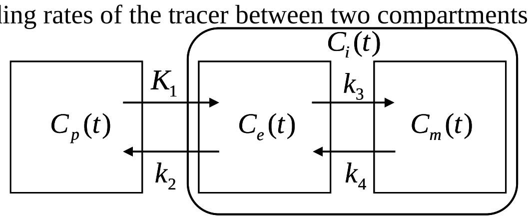


Fig.1. 3-Compartment model of tissue in FDG PET inspection [2], [4] with four tracer's flow parameters K_1, k_2, k_3, k_4.

$$\begin{cases} C_i(t) = C_e(t) + C_m(t) \\ \dfrac{dC_e(t)}{dt} = K_1 C_p(t) - (k_2 + k_3) C_e(t) + k_4 C_m(t) \\ \dfrac{dC_m(t)}{dt} = k_3 C_e(t) - k_4 C_m(t) \end{cases} \quad (1)$$

In this model, the input function is $C_p(t)$, and the output is $C_i(t)$. Let $\otimes$ denote the convolution, and let g(t) be the impulse response of the compartment model described by Fig. 1 and Eq. (1). Then the output function $C_i(t)$

including the flow parameters is described by the following Eq. (2) and Eq. (3) ([2], [4]).

$$C_i(t) = g(t) \otimes C_p(t) = \int_0^t g(t-\tau) C_p(\tau) d\tau \qquad (2)$$

where

$$\begin{cases} g(t) = \dfrac{K_1}{\beta - \alpha}\left[(k_3 + k_4 - \alpha) e^{-\alpha t} + (\beta - k_3 - k_4) e^{-\beta t} \right] \\ \alpha, \beta = \dfrac{1}{2}\left\{ (k_2 + k_3 + k_4) \mp \sqrt{(k_2 + k_3 + k_4)^2 - 4 k_2 k_4} \right\} \end{cases} \qquad (3)$$

In this PET inspection, the problem is to estimate the values of flow parameters K_1, k_2, k_3, k_4 in the impulse response $g(t)$ from the two observed time series data, C_p(t) and C_i(t), where $C_i(t)$ is normally measured as a noisy time series data $r(t)$ (=$C_i(t)$+noise) by PET Camera.

2.2 Estimation Using Linear Regression Analysis and DCE.

We equivalently transform the Eq. (1) to a linear regression model (LRM) equation using cumulative function, based on the following Theorem 1 ([9]) and Theorem 2 of DCE [12] that is derived from Theorem 1.

Theorem 1 [9]:

Let $f(x,t)$ and $\partial f(x,t)/\partial t$ be a continuous function and its continuous partial derivative function over the closed section [a,b], respectively. And let the a and b are the functions $a(t)$ and $b(t)$ with respect to t, respectively. Then, Eq. (4) holds.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)dx + f(b(t),t)\frac{db(t)}{dt} - f(a(t),t)\frac{da(t)}{dt} \qquad (4)$$

Theorem 2 (DCE: Differentiation of Convolution with Exponential function) [12]:

Let $J(t)$ be the convolution between the functions $exp(-kt)$ and differentiable continuous function $C(t)$ as shown in Eq. (5), then from Theorem 1, Eq. (6) holds.

$$\frac{dJ(t)}{dt} \equiv J'(t) = (-k)J(t) + C(t) \qquad (5)$$

$$J(t) \triangleq e^{-kt} \otimes C(t) = \int_0^t e^{-k(t-\tau)} C(\tau) d\tau \qquad (6)$$

Let $r(t)$ be the measured value by PET camera at time t. The r(t) normally includes the additive noise $n(t)$, so $r(t)=C_i(t) + n(t)$. However, for the simplicity, we simply assume the case where the noise is not included in the measured value, then $r(t)=C_i(t)$.

Then, from Theorem 2, we obtain the Eq. (7).

$$r''(t) = (k_3 + k_4) K_1 C_p(t) + K_1 C'_p(t) - (k_2 + k_3 + k_4) r'(t) - k_2 k_4 r(t) \qquad (7)$$

where

$r'(t) = dr(t)/dt,\ r''(t) = d^2 r(t)/dt^2,\ C'_p(t) = dC_p(t)/dt.$

Integrating both sides of Eq. (7) twice with respect to time t, we have the following Eq. (8) using cumulative functions such as

$$r(t) = AC_p^{II}(t) + BC_p^{I}(t) + Cr^{I}(t) + Dr^{II}(t) \qquad (8)$$

where

$$\begin{aligned} C_p^I(t) &\triangleq \int_0^t C_p(\tau)d\tau,\ C_p^{II}(t) \triangleq \int_0^t C_p^I(\tau)d\tau \\ r^I(t) &\triangleq \int_0^t r(\tau)d\tau,\ r^{II}(t) \triangleq \int_0^t r^I(\tau)d\tau \end{aligned} \qquad (9)$$

$$\begin{cases} A = K_1(k_3 + k_4),\ B = K_1 \\ C = -(k_2 + k_3 + k_4),\ D = -k_2 k_4 \end{cases} \qquad (10)$$

Eq. (8) can be regarded as a linear regression equation with objective variable $r(t)$ and four explanatory variables such as $C_p^{II}(t), C_p^I(t), r^I(t), r^{II}(t)$. Since the time series data $r(t)$ and $C_p(t)$ are given by PET Camera and blood sampling respectively, it is possible to obtain the values of parameters A, B, C, and D of Eq. (8), by linear regression analysis ([10], [11]). Then we can estimate the parameters K_1, k_2, k_3, and k_4 from Eq. (11).

This is the main idea of the proposed method, that is, we transform the simultaneous differential equations that describes the system operating characteristics into an equivalent equation that is convenient for linear regression analysis.

2.3 Weighted LRM-DCE Method.

Generally, in many cases, it is expected that noise will be mixed into the parameter estimation. For such occasion, we propose the Weighted LRM-DCE method that multiplies a known function w(t) to the both sides of Eq. (8) as weight function, as shown in Eq. (11).

$$r(t)w(t) = AC_p^{II}(t)w(t) + BC_p^I(t)w(t) + Cr^I(t)w(t) + Dr^{II}(t)w(t) \qquad (11)$$

To determine the parameters A, B, C, and D in Eq. (9) is equivalent to obtain the same parameters in Eq. (8). This weighting function $w(t)$ has to be appropriately selected according to the noise characteristics.

For example, as the weighting function w(t), we propose $r(t)^{-1}$ that is the inverse of the observed r(t) in the region where $r(t)>0$. Then we have the following Eq. (12).

$$1 = AC_p^{II}(t)r(t)^{-1} + BC_p^I(t)r(t)^{-1} + Cr^I(t)r(t)^{-1} + Dr^{II}(t)r(t)^{-1} \qquad (12)$$

3. Applied Experimentation

3.1 Programming Environment for Experiments

The programming environment for numerical computation used in the experimentation are as follows.

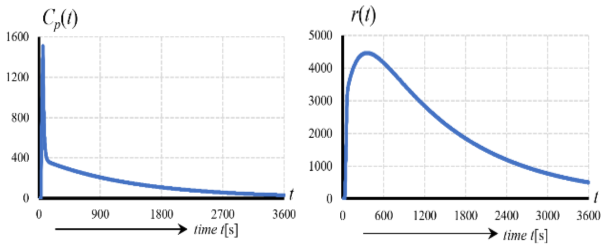
CPU: Intel Core i7-3770 (3.40GHz), RAM: 16GB,
OS: Windows 10 Professional (64bit),
Programming Language: Borland C++ 5.51 (32bit).

3.2 Experiments for PET compartment Model

We have experimented for an input data $C_p(t)$ (Fig. 2 (a)) that is generated based on the literature [6] and noiseless output data $r(t)$ (Fig. 2 (b)) generated from the parameters $(K_1, k_2, k_3, k_4) = (0.200, 0.130, 0.060, 0.007)$ using Eq. (2) and Eq. (3). Moreover, we also experimented for $r(t)$ (Fig. 3 (a)) where the noise $n(t)$~N(0,1002) is added to the aforementioned noiseless r(t). In this noisy case, as a matter of fact, we used the smoothed r(t) (Fig.3 (b)) where moving average processing is done twice over the range of ±4 points before and after each time point of the noisy $r(t)$.

For the evaluation of estimation accuracy, we define the relative error as (|(Estimated Value)-(True Value)| / (True Value))× 100[%].

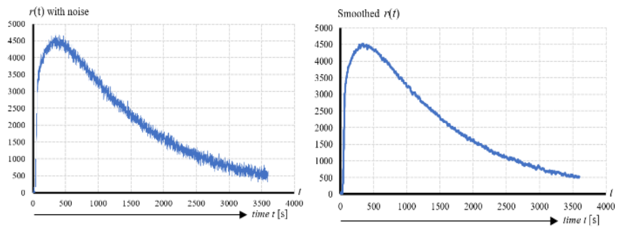
A. Noiseless Case



(a) $C_p(t)$ (b) *noiseless* $r(t)$

Fig.2. Generated input $C_p(t)$ and noiseless $r(t)$.

B. Noisy Case



(a) noisy $r(t)$ (b) smoothed $r(t)$

Fig.3. Generated noisy $r(t)$ and smoothed $r(t)$.

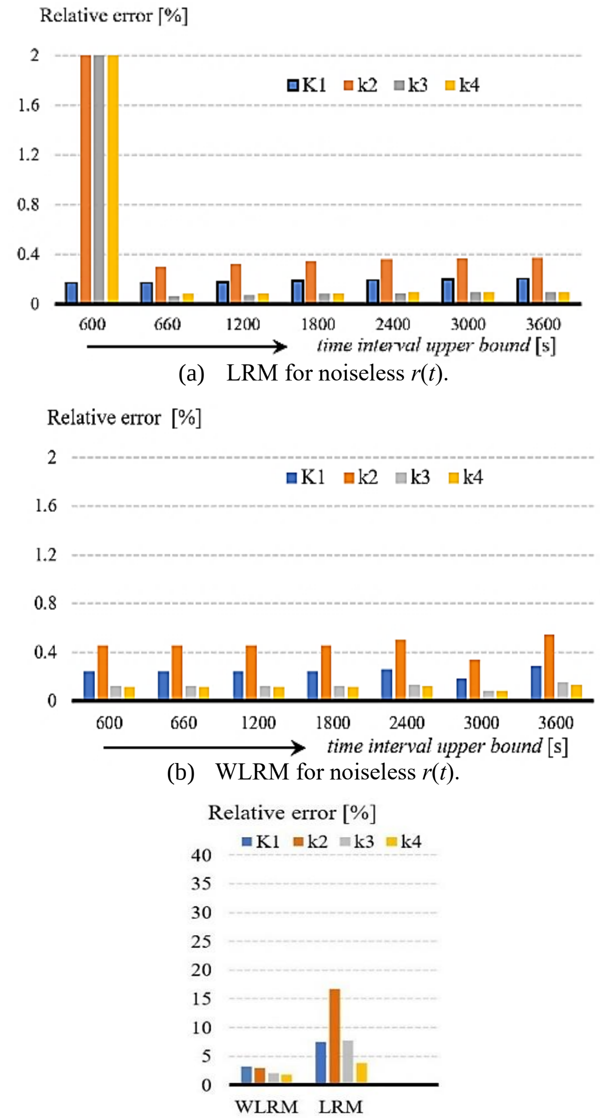
C. Results

We briefly call the LRM-DCE method and Weighted LRM-DCE method as LRM and WLRM, respectively.

The estimated results of (K_1, k_2, k_3, k_4) by both LRM and WLRM are shown in Fig. 4 for each parameter in each time interval.

Fig.4 (a) shows that the relative error rate (RER) of parameters estimation by LRM becomes less than 0.4 [%] for noiseless $r(t)$, if we take the time series data of 660 seconds (11 minutes) or longer for estimation computing. Fig. 4(b) shows that the resultant (RER) by WLRM is less than 0.46[%] until 1800[sec]. Even in the whole-time interval (=3600[sec]), the RER is less than 0.55[%]. Fig. 4(c) illustrates the resultant RER for noisy $r(t)$ by both WLRM and LRM, where WLRM shows that the rate is less than 3.3 [%] for all of four parameters in the whole-time interval, however the RER of LRM is worse than WLRM.

As for the processing time for both noisy and noiseless $r(t)$ by WLRM and LRM in the whole-time interval, the computing time of both method is less than 80 [ms].



(a) LRM for noiseless $r(t)$.

(b) WLRM for noiseless $r(t)$.

(c) WLRM and LRM for nosy $r(t)$.

Fig. 4. Accuracy of the parameter estimation for K_1, k_2, k_3, and k_4 by the methods of LRM and WLRM ([12]).

3.3 Experiments for RLC Series Electrical Circuit

A. Equation and the Impulse Response

RLC series electrical circuit (RLC-EC) is one of most simple compartment model, that consists of three basic passive elements connected in series: a resistor R, an inductor L, and a capacitor C. Eq. (13) shows the system equation of RLC-EC with DC power supply (DC-PS) E (Fig. 5(a)), where we can regard it as the form of LRM itself without DCE processing.

$$L\frac{di(t)}{dt}+Ri(t)+\frac{1}{C}\int_0^t i(\tau)d\tau=E \tag{13}$$

Then, we have the following Eq. (14) for parameters estimation, using cumulative functions.

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$$Li(t)+Ri^{I}(t)+\frac{1}{C}i^{II}(t)=\int_0^t Edt$$

where

$$\begin{cases} i^{I}(t)\triangleq\int_0^t i(\tau)d\tau \\ i^{II}(t)\triangleq\int_0^t i^{I}(\tau)d\tau=\int_0^t\int_0^\tau i(\xi)d\xi d\tau \end{cases} \quad (14)$$

Considering the RLC series circuit as a system of compartment model, the parameters of impulse response are those values of R[Ω], L[H], C[F], themselves. And, there are three types of impulse responses, depending on the magnitude relationship between $(R/2L)^2$ and $1/LC$.

Let those impulse responses be $g_1(t)$, $g_2(t)$, $g_3(t)$, depending on the cases, $(R/2L)^2 < 1/LC$, $(R/2L)^2 = 1/LC$, $(R/2L)^2 > 1/LC$, respectively.

(i) Oscillatory case: $(R/2L)^2 < 1/LC$

$$g_1(t)=\frac{1}{\sqrt{\left(\frac{L}{C}\right)-\left(\frac{R}{2}\right)^2}}\exp\left(-\frac{R}{2L}t\right)\cdot\sin\left(\sqrt{\left(\frac{1}{LC}\right)-\left(\frac{R}{2L}\right)^2}\right)t$$

(ii) Critical damping case: $(R/2L)^2 = 1/LC$

$$g_2(t)=\frac{1}{L}t\cdot\exp\left(-\frac{R}{2L}t\right)$$

(iii)Over-damping case: $(R/2L)^2 > 1/LC$

$$g_3(t)=\frac{1}{\sqrt{\left(\frac{R}{2}\right)^2-\left(\frac{L}{C}\right)}}\exp\left(-\frac{R}{2L}t\right)\cdot\sinh\left(\sqrt{\left(\frac{R}{2L}\right)^2-\left(\frac{1}{LC}\right)}\right)t$$

Fig. 5 shows the graphs of RLC circuit with DC-PS E and cumulative functions of $i(t)$ in the case of oscillatory case.

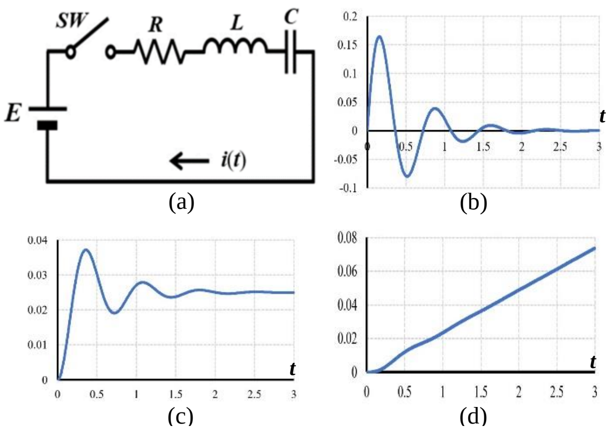

Fig.5. (a) RLC series electrical circuit with DC-PS E. (b) The $i(t)$ in the oscillatory case. (c) $i^{I}(t)$. (d) $i^{II}(t)$.

B. Results of RLC parameters estimation

We have experimented for the parameters estimation, from generated time series data of 3 seconds current $i(t)$ by taking 0.003[s] increments, that consists of 1001 pieces from time t=0 to t=3.

Estimated results (ER) and RER for each of above three cases with E=5[V] are as in the followings.

(i) Oscillatory case: (R, L, C) = (10, 2.5, 0.005)
ER: (10.00120, 2.49988, 0.00499), RER < 0.2%.

(ii) Critical damping case: (R, L, C) = (10, 2.5, 0.1)
ER: (10.00010, 2.50001, 0.10000), RER < 0.01%.

(iii) Over-damping case: (R, L, C) = (10, 2.5, 0.2)
ER: (10.00000, 2.50003, 0.19999), RER <0.01%.

4. Conclusion

In this paper, we have proposed a method for parameters estimation of impulse response in compartment model systems. The proposed idea is based on the equivalent transformation from the simultaneous differential equations of the system behavior into an equation that is suitable for the estimation by linear regression analysis.

For further study, we are going to experiment for more complicated circuits and pursue the noise cancelation method.

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