Synchronization of Novel 5D Hyperchaotic Systems

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Abstract

In this paper, synchronization of novel five-dimensional (5D) autonomous hyperchaotic systems is studied. The synchronization control law is proposed based on the center translation method. A structure compensator is formulated to make the mathematical model of the error system the same as that of the response system, and a linear feedback controller is designed via the Lyapunov stability theory to make the error system globally asymptotically stable at the origin. Thus, the two 5D hyperchaotic systems are synchronized. Some relevant numerical simulation results, such as the curves of the corresponding synchronization state variables and the errors, are given to illustrate the feasibility and effectiveness of the synchronization control law.

Keywords: novel 5D hyperchaotic system, hyperchaos synchronization, center translation method, Lyapunov stability theory, global asymptotic stability

1. Introduction

Hyperchaos was first presented in 1979 by Otto Rössler.[1] Because hyperchaos is much more complicated than chaos, hyperchaos synchronization has greater application significance and engineering value in secure communication.

In this paper, the mathematical model of the novel 5D hyperchaotic system is given as the drive system. Hyperchaos synchronization of the 5D systems is studied based on the center translation method. Corresponding numerical simulation results are presented to demonstrate the validity of the synchronization method.

2. The Novel 5D Hyperchaotic System

The dynamic equations of the novel 5D hyperchaotic system are

$$\begin{aligned}
\dot{x}_1 &= a\left(y_1 - x_1\right), \\
\dot{y}_1 &= \left(c - a\right)x_1 + cy_1 + w_1 - x_1 z_1, \\
\dot{z}_1 &= -bz_1 + x_1 y_1, \\
\dot{v}_1 &= mw_1, \\
\dot{w}_1 &= -y_1 - hv_1,
\end{aligned} \tag{1}$$

where $x_1, y_1, z_1, v_1, w_1 \in R$ are state variables, and a = 23, b = 3, c = 18, m = 12 and h = 4.[2]

Let the initial values of the system (1) be (x_{10}, y_{10}, z_{10}, v_{10}, w_{10}) = (1, 1, 1, 1, 1), then the Lyapunov exponents respectively are $\lambda_{11} = 0.8732 > 0$, $\lambda_{12} = 0.1282 > 0$, $\lambda_{13} = -0.0013 \approx 0$, $\lambda_{14} = -0.5770 < 0$ and $\lambda_{15} = -8.4231 < 0$. It indicates that the system (1) is hyperchaotic. The attractors of the 5D hyperchaotic system (1) are shown in Fig. 1.

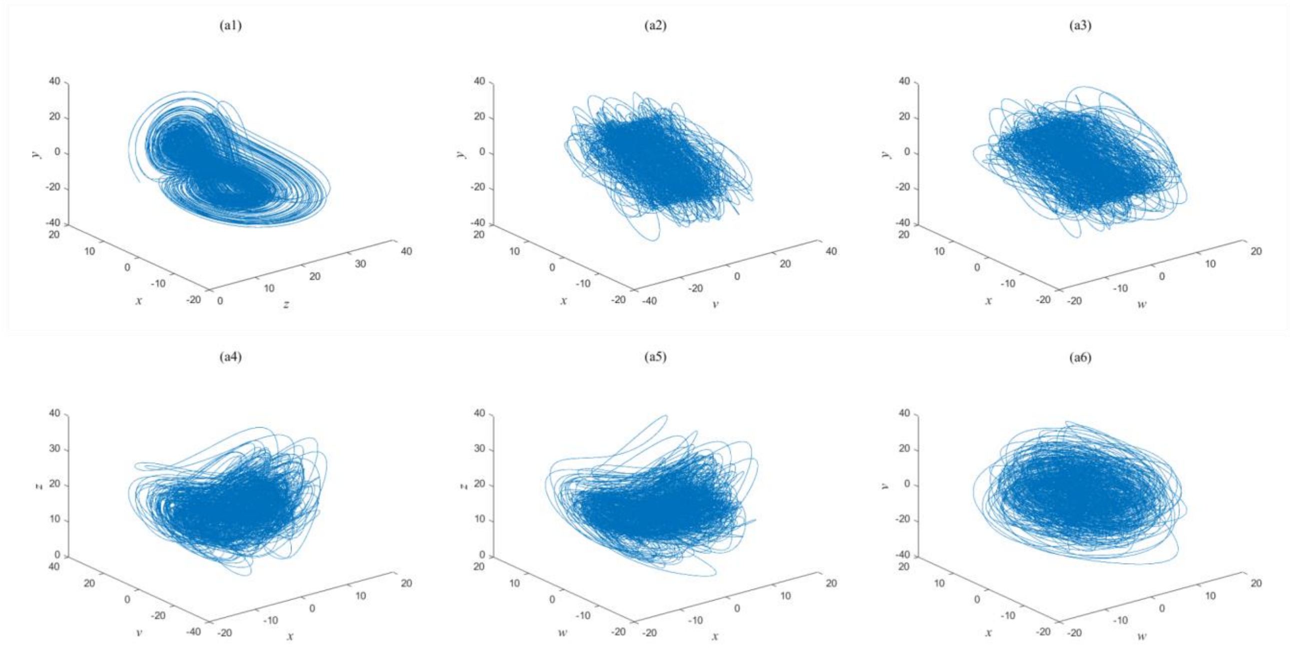


Fig. 1. Attractors of the 5D hyperchaotic system: (a1) *z*-*x*-*y*; (a2) *v*-*x*-*y*; (a3) *w*-*x*-*y*; (a4) *x*-*v*-*z*; (a5) *x*-*w*-*z*; (a6) *w*-*x*-*v*

3. Hyperchaos Synchronization Based on Center Translation Method

3.1. *Formulation of error system*

Take the system (1) as the drive system, then the response system is formulated as

$$\begin{aligned}
\dot{x}_2 &= a\left(y_2 - x_2\right) + u_{s1} + u_{c1},\\
\dot{y}_2 &= \left(c-a\right)x_2 + cy_2 + w_2 - x_2 z_2 + u_{s2} + u_{c2},\\
\dot{z}_2 &= -bz_2 + x_2 y_2 + u_{s3} + u_{c3},\\
\dot{v}_2 &= mw_2 + u_{s4} + u_{c4},\\
\dot{w}_2 &= -y_2 - hv_2 + u_{s5} + u_{c5},
\end{aligned} \qquad (2)$$

where

$$\boldsymbol{u}_s = \begin{bmatrix} u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5} \end{bmatrix}^{\mathrm{T}}$$

and

$$\boldsymbol{u}_c = \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5} \end{bmatrix}^{\mathrm{T}}$$

are structure compensator and synchronization controller to be designed. Let $\boldsymbol{u}_s = \mathbf{0}$, $\boldsymbol{u}_c = \mathbf{0}$, and the initial values of the response system (2) be (x_{20}, y_{20}, z_{20}, v_{20}, w_{20}) = (5, 0, 4, 3, 8), then the Lyapunov exponents respectively are $\lambda_{21} = 0.9121 > 0$, $\lambda_{22} = 0.1175 > 0$, $\lambda_{23} = -0.0008 \approx 0$, $\lambda_{24} = -0.5533 < 0$ and $\lambda_{25} = -8.4755 < 0$. It shows that the response system (2) is also hyperchaotic.

Let

$$\begin{aligned}
\boldsymbol{e} &= \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{bmatrix}^{\mathrm{T}}\\
&= \begin{bmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 & v_2 - v_1 & w_2 - w_1 \end{bmatrix}^{\mathrm{T}}
\end{aligned}$$

be the synchronization error and

$$\begin{aligned}
\boldsymbol{u}_s &= \begin{bmatrix} u_{s1} & u_{s2} & u_{s3} & u_{s4} & u_{s5} \end{bmatrix}^{\mathrm{T}}\\
&= \begin{bmatrix} 0 \\ x_2 z_1 + x_1 z_2 - 2x_1 z_1 \\ -x_2 y_1 - x_1 y_2 + 2x_1 y_1 \\ 0 \\ 0 \end{bmatrix},
\end{aligned}$$

then the error system is simplified as

$$\begin{aligned}
\dot{e}_1 &= a\left(e_2 - e_1\right) + u_{c1},\\
\dot{e}_2 &= \left(c-a\right)e_1 + ce_2 + e_5 - e_1 e_3 + u_{c2},\\
\dot{e}_3 &= -be_3 + e_1 e_2 + u_{c3},\\
\dot{e}_4 &= me_5 + u_{c4},\\
\dot{e}_5 &= -e_2 - he_4 + u_{c5}.
\end{aligned} \qquad (3)$$

Comparing the mathematical model of the error system (3) with that of the controlled system (2) in Ref.

2, it can be found that the two models are similar. Hence, the synchronization controller $\boldsymbol{u}_c$ is designed as

$$\boldsymbol{u}_c = \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5} \end{bmatrix}^{\mathrm{T}}$$
$$= \begin{bmatrix} -k_1 e_1 & -k_2 e_2 & -k_3 e_3 & -k_4 e_4 & -k_5 e_5 \end{bmatrix}^{\mathrm{T}},$$

where $k_1, k_2, k_3, k_4, k_5 \geq 0$.

3.2. *Design of linear feedback synchronization controller*

Theorem 1. *Let* $\boldsymbol{x} = \boldsymbol{0}$ *be an equilibrium point for* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, *where* $\boldsymbol{f}: D \to R^n$ *is a locally Lipschitz map from a domain* $D \subset R^n$ *into* R^n. *Let* $V: R^n \to R$ *be a continuously differentiable function such that*

$$V(\boldsymbol{0}) = 0 \quad \text{and} \quad V(\boldsymbol{x}) > 0, \quad \forall\, \boldsymbol{x} \neq \boldsymbol{0}$$
$$\|\boldsymbol{x}\| \to \infty \Rightarrow V(\boldsymbol{x}) \to \infty$$
$$\dot{V}(\boldsymbol{x}) < 0, \quad \forall \boldsymbol{x} \neq \boldsymbol{0}$$

then $\boldsymbol{x} = \boldsymbol{0}$ *is globally asymptotically stable.*[2]

Take a continuously differentiable function

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + \frac{h}{m} e_4^2 + e_5^2 \right)$$

as a Lyapunov function candidate for the error system (3). Then, the derivative $\dot{V}$ is derived as

$$\begin{aligned}
\dot{V} &= e_1\dot{e}_1 + e_2\dot{e}_2 + e_3\dot{e}_3 + \frac{h}{m} e_4\dot{e}_4 + e_5\dot{e}_5 \\
&= -(k_1 + a)e_1^2 + c e_1 e_2 - (k_2 - c)e_2^2 \\
&\quad -(k_3 + b)e_3^2 - k_4 \frac{h}{m} e_4^2 - k_5 e_5^2 \\
&\leq -\left(k_1 + a - \frac{c}{2}\right)e_1^2 - \left(k_2 - \frac{3}{2}c\right)e_2^2 \\
&\quad -(k_3 + b)e_3^2 - k_4 \frac{h}{m} e_4^2 - k_5 e_5^2.
\end{aligned}$$

For $\dot{V} < 0$, the parameters k_1, k_2, k_3, k_4 and k_5 should satisfy that

$$\begin{matrix} k_1 + a - \frac{c}{2} > 0, \\ k_2 - \frac{3}{2}c > 0, \\ k_3 + b > 0, \\ k_4 \frac{h}{m} > 0, \\ k_5 > 0, \end{matrix} \;\Rightarrow\; \begin{matrix} k_1 > \frac{c}{2} - a, \\ k_2 > \frac{3}{2}c, \\ k_3 > -b, \\ k_4 > 0, \\ k_5 > 0, \end{matrix} \;\Rightarrow\; \begin{matrix} k_1 = 0, \\ k_2 = 30, \\ k_3 = 0, \\ k_4 = 1, \\ k_5 = 1. \end{matrix}$$

Thus, the linear feedback synchronization controller $\boldsymbol{u}_c$ is designed as

$$\boldsymbol{u}_c = \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} & u_{c5} \end{bmatrix}^{\mathrm{T}}$$
$$= \begin{bmatrix} 0 & -30e_2 & 0 & -e_4 & -e_5 \end{bmatrix}^{\mathrm{T}}.$$

From Theorem 1, the error system (3) is globally asymptotically stable at the origin. It indicates that the response system (2) is synchronized with the drive system (1).

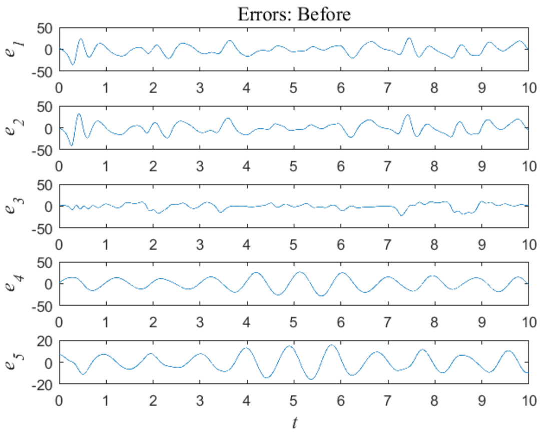
3.3. *Numerical simulation*

Remark 1. *The initial values of the drive system (1) and the response system (2) are* (x_{10}, y_{10}, z_{10}, v_{10}, w_{10}) = (1, 1, 1, 1, 1) *and* (x_{20}, y_{20}, z_{20}, v_{20}, w_{20}) = (5, 0, 4, 3, 8) *respectively in this paper.*

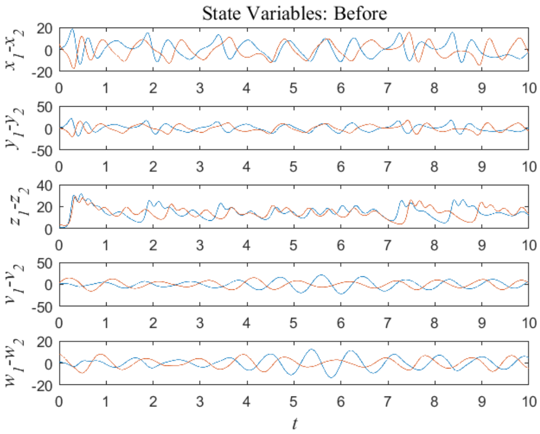
Definition 1. *After adding the structure compensator* $\boldsymbol{u}_s$ *and the linear feedback synchronization controller* $\boldsymbol{u}_c$ *to the response system (2), the Lyapunov exponents of the response system (2) are called sub-Lyapunov exponents.*[3]

Theorem 2. *The response system (2) and the drive system (1) will synchronize only if the sub-Lyapunov exponents are all negative.*[3]

The curves of the errors and the corresponding state variables before and after adding the structure compensator $\boldsymbol{u}_s$ and the linear feedback synchronization controller $\boldsymbol{u}_c$ to the response system (2) are shown in Fig. 2 and Fig. 3 respectively. Comparing Fig. 3 with Fig. 2, it can be found that the errors e_1, e_2, e_3, e_4 and e_5 converge to zero asymptotically and rapidly and the corresponding state variables are synchronized well after adding $\boldsymbol{u}_s$ and $\boldsymbol{u}_c$ to the response system (2). Moreover, the sub-Lyapunov exponents of the response system (2) are λ_{21c} = -1.0292, λ_{22c} = -1.0355, λ_{23c} = -3.0000, λ_{24c} = -17.4669 and λ_{25c} = -17.4690, which are all negative. From Theorem 2, the response system (2) and the drive system (1) have synchronized.

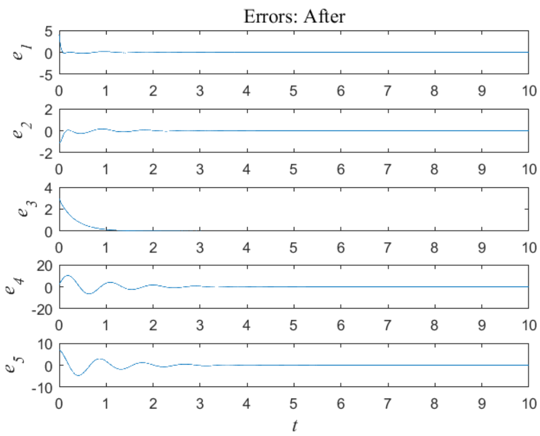


(a) Errors

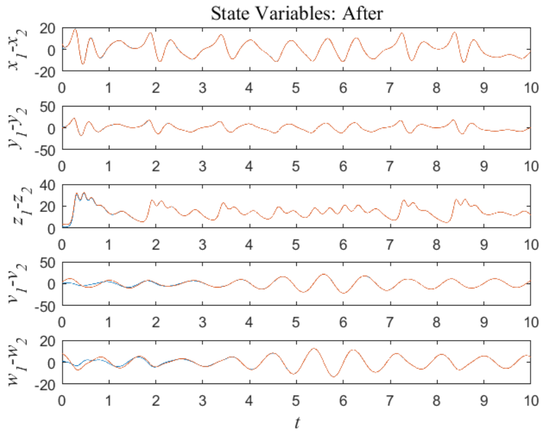


(b) State variables

Fig. 2. Before: (a) Errors; (b) State variables



(a) Errors



(b) State variables

Fig. 3. After: (a) Errors; (b) State variables

4. Conclusions

Synchronization of the novel 5D hyperchaotic systems is proposed based on the center translation method in this paper. Numerical simulation results illustrate the feasibility of the synchronization method. The study has some engineering significance. Furthermore, the circuit implementation of the synchronization system is under investigation and will be reported elsewhere.

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Author Introduction

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