Consensus Control of Linear Discrete-time Multi-agent Systems with Limited Communication Data Rate

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Abstract

This paper investigates the consensus problem of linear discrete-time multi-agent systems with limited communication data rate and the cooperative-antagonistic interactions. A consensus control protocol is designed based upon a dynamic encoding-decoding mechanism. By means of the proposed control protocol, it is guaranteed that the agents can attain bipartite consensus if the signed undirected graph is connected and structurally balanced/unbalanced. Moreover, the clear form of the convergence rate is given.

Keywords: Multi-agent systems, signed undirected graph, bipartite consensus, structural balance, encoding and decoding

1. Introduction

Consensus is one of the hottest topics in the field of multi-agent systems (MASs) due to its wide application in lighting. Many important and interesting problems about consensus have been studied in recent years.

In many relevant works, it's assumed that the agents can achieve exact information of the states of the neighbors by local communication. However, in practice, the communication ability of the network is limited. Thus, the encoding and decoding scheme is introduced to deal with this problem[1-3].

In addition, most of the works in the literature assume that the relationship between agents is cooperative. However, cooperation often coexists with antagonism. By utilizing the property of structural balance/unbalance of communication networks, the bipartite consensus is introduced where all gents attain agreement concerning a value, which is same for all in modulus but not in sign[4-5].

Motivated by the above discussion, we consider a consensus control protocol based on a dynamic encoding-decoding mechanism. The agents can achieve bipartite consensus under the control scheme. What's more, the clear form of the convergence rate is given.

2. Preliminaries and Problem Formulation

2.1. *Signed Graph*

Let $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$ be a signed undirected graph, where $\mathcal{V}=\{v_1,v_2,\cdots,v_N\}$ is a set of N agents with i denoting the ith agent, $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ is the edge set of agents, and $\mathcal{A}=[a_{ij}]\in\mathbb{R}^{N\times N}$ with the signed elements is the weighted adjacency matrix of $\mathcal{G}$. The set of agents who

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can communicate with agent i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} \mid (i,j) \in \mathcal{E}\}$. The Laplacian matrix of $\mathcal{G}$ is represented by $\mathcal{L} = \mathcal{C} - \mathcal{A}$, where $\mathcal{C} = diag(\sum_{j\in\mathcal{N}_1}|a_{1j}|, \cdots, \sum_{j\in\mathcal{N}_N}|a_{Nj}|)$. A signed graph $\mathcal{G}$ is structurally balanced if the nodes in $\mathcal{V}$ are divided into two parts $\{\mathcal{V}_1, \mathcal{V}_2\}$, where $\mathcal{V}_1 \bigcup \mathcal{V}_2 = \mathcal{V}$ and $\mathcal{V}_1 \bigcap \mathcal{V}_2 = \varnothing$, bring $a_{ij} \geq 0 \quad \forall i, j \in \mathcal{V}_p\left(p \in \{1,2\}\right)$, $a_{ij} \leq 0 \ \forall v_i \in \mathcal{V}_p, v_j \in \mathcal{V}_q, \ p \neq q \ \left(p, q \in \{1,2\}\right)$, otherwise, the signed graph $\mathcal{G}$ is structurally unbalanced.

2.2. *Problem Formulation*

In this paper, we consider the bipartite consensus control for a multi-agent system with the discrete-time dynamics

$$x_i(k+1) = Ax_i(k) + Bu_i(k) \tag{1}$$

Where $x_i(k) \in \mathbb{R}^n$ and $u_i(k) \in \mathbb{R}$, respectively, represent the state and input of the agent i at time k. $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times 1}$ denote the state and input matrix.

A uniform quantizer is utilized to quantize the state of agent. The quantizer $q(x)$ is defined as follows:

$$q(x) = \begin{cases} 0, & -1/2 \leq x \leq 1/2 \\ b, & (2b-1)/2 < x \leq (2b+1)/2, \\ & b = 1, \cdots, L-1 \\ L, & x > (2L-1)/2 \\ -q(-x), & x < -1/2 \end{cases} \tag{2}$$

Obviously, the level of the quantizer is $2L+1$. When $|x| \leq L + 1/2$, the quantization error of the uniform quantizer satisfies $|x - q(x)| \leq 1/2$. Assume that the information transmitted between agents only can be binary number, but not exact value. Thus, the corresponding encoder and decoder are designed. The encoder of agent i is proposed as follows:

$$\begin{cases} \hat{\phi}_i(k) = l(k-1)Z_i(k) + A\hat{\phi}_i(k-1), \hat{\phi}_i(0) = 0 \\ Z_i(k) = Q(\dfrac{x_i(k) - A\hat{\phi}_i(k-1)}{l(k-1)}) \end{cases} \tag{3}$$

Where $\hat{\phi}_i(k)$ and $Z_i(k)$ are the internal variable and output of the encoder, respectively. Apart from this, $Q(\cdot) = [q(\cdot), \cdots, q(\cdot)]^T$ is the product quantizer. $l(k) = l_0 r^k$ is the scaling function, where $l_0 \in \mathbb{R}$ and $r \in (0,1)$, which prevents the quantizer from saturation. The decoder of agent j which receive the information from agent i is designed as follows:

$$\begin{cases} \hat{\varphi}_{ji}(0) = 0 \\ \hat{\varphi}_{ji}(k) = l(k-1)Z_i(k) + A\hat{\varphi}_{ji}(k-1) \end{cases} \tag{4}$$

Where $\hat{\varphi}_{ji}(k)$ which is connected with agent i is the internal variable of the decoder of agent j.

The objective of this paper is to propose a distributed control scheme based on dynamic encoding and decoding, such that the MASs achieve bipartite consensus when communicate graph is structurally balanced/unbalanced.

For this purpose, we consider the following protocol

$$u_i(k) = K\sum_{j\in N_i}|a_{ij}|(\hat{\phi}_i(k) - \text{sgn}(a_{ij})\hat{\varphi}_{ij}(k)) \tag{5}$$

Where $K \in \mathbb{R}^{n\times 1}$ is the control gain and $\text{sgn}(\cdot)$ represents the signum function.

In order to facilitate the subsequent analysis, we give the following definition and assumption.

Definition 1. *The consensus of Multi-agent systems can be achieved, if there exists the control law such that*

$$\lim_{k\to\infty}\|x_i(k) - \text{sgn}(a_{ij})x_j(k)\| = 0$$

For any initial state $x_i(0)$, $\forall i \in [1, N]$.

Assumption 1. $\|x_i(0)\|_\infty \leq C_x, \forall i \in [1, N]$ *and* C_x *can be achieved by all agents.*

3. Main Results

3.1. *Structurally balanced graph*

To get the main result, we need divide the agents into two groups $d_i \in \{1, -1\}$ $i = 1, \cdots, N$, where the agents are in the same group when they are cooperating, i.e., $a_{ij} > 0$.

The relevant lemmas are given as follows:

Lemma 1. *For a connected signed and structurally balanced graph, there always exists a diagonal matrix* $D = diag(d_1, \cdots, d_N)$ *such that all elements of* $D\mathcal{A}D$ *is nonnegative, and* $0 = \lambda_1(\mathcal{L}) < \lambda_2(\mathcal{L}) \leq \cdots \leq \lambda_N(\mathcal{L})$.

Lemma 2. *For any* $P \in \mathbb{R}^{n\times n}$ *and* $\varepsilon > 0$, *we can obtain that* $\|P^k\| \leq M\eta^k, \forall k \geq 0$, *where* $M = \sqrt{n}(1 + 2/\varepsilon)^{n-1}$ *and* $\eta = \rho(P) + \varepsilon\|P\|$.

Theorem 1. *Consider the multi-agent system (1) with the structurally balanced and connected signed undirected communication network. Suppose that* (A, B) *is controllable,* $\prod_j|\lambda_j(A)| < (1 + \lambda_2/\lambda_N)/(1 - \lambda_2/\lambda_N)$ *and Assumption 1 holds. Select appropriate control gain* K *such that* $\rho(J(K)) < 1$, *where* $J(K) = diag(A, A + \lambda_2 BK, \cdots, A + \lambda_N BK)$. *For any* $r \in (\rho(J(K)), 1)$ *and* $\sigma \in (0, (r - \rho(J(K)))/\|J(K)\|)$, *let*

$$M(K, r) = \frac{\|A\|_\infty + 2d^*\|BK\|_\infty}{2r} + \frac{n^{3/2}\lambda_N^2 M\sqrt{N}\|BK\|_\infty^2}{r^2(1-\eta)}$$

and for any $L > M(K, r) - 1/2$, *let*

$$l_0 > \max\{\frac{\|A\|_\infty C_x}{L+1/2}, \frac{4r^2 C_x(1-\eta)}{\sqrt{n}\lambda_N \|BK\|_\infty}\}$$

Then under the proposed control scheme (5) given by (2), (3) and (4), the closed-loop system satisfies the bipartite average consensus

$$\lim_{k\to\infty} x_i(k) = d_i \sum_{j=1}^N d_j x_j(0),\ i = 1,\cdots,N$$

Proof. Let the signed average state error be $\delta_i(k) \triangleq x_i(k) - d_i x_{ave}(k)$, where $x_{ave}(k) \triangleq 1/N\sum_{i=1}^N d_i x_i(k)$ is the signed average state of the MASs. It yields that

$$\lim_{k\to\infty}\|\delta_i(k)\| \le 1/N\sum_{j=1}^N \lim_{k\to\infty}\|x_i(k) - \text{sgn}(a_{ij})x_j(k)\| \quad (6)$$

Simultaneously, $\lim_{k\to\infty}\|\delta_i(k)\| = 0$ also can implies the consensus of the MAS. Thus, the bipartite consensus of structurally balanced network is equivalent to $\lim_{k\to\infty}\|\delta_i(k)\| = 0$, for all $i \in [1, N]$.

According to the encoder and decoder, it can be achieved that $\hat{\varphi}_{ij}(k) \equiv \hat{\phi}_j(k)$, $j \in \mathcal{N}_i$, $\forall i \in [1,N]$ easily. Let the estimation error be $e_i(k) = x_i(k) - \hat{\phi}_i(k)$. Inserting the controller into the linear discrete system, we have

$$x_i(k+1) = Ax_i(k) + BK\sum_{j=1}^N |a_{ij}|(x_i(k) - \text{sgn}(a_{ij})x_j(k)) - BK\sum_{j=1}^N |a_{ij}|(e_i(k) - \text{sgn}(a_{ij})e_j(k)) \quad (7)$$

Choose $X(k) = [x_1^T(k),\cdots,x_N^T(k)]^T$, $E(k) = [e_1^T(k),\cdots,e_N^T(k)]^T$ and $\Delta(k) = [\delta_1^T(k),\cdots,\delta_N^T(k)]^T$. Then

$$X(k+1) = (I_N \otimes A + \mathcal{L}\otimes BK)X(K) - (\mathcal{L}\otimes BK)E(K) \quad (8)$$

$$\Delta(K+1) = (I_N \otimes A + \mathcal{L}\otimes BK)\Delta(K) - (\mathcal{L}\otimes BK)E(K) \quad (9)$$

Denote $\Phi(k) = [\phi_1^T(k),\cdots,\phi_N^T(k)]^T$. Then

$$X(k+1) - (I_N \otimes A)\Phi(k) = (I_N \otimes A - \mathcal{L}\otimes BK)E(k) + (\mathcal{L}\otimes BK)\Delta(k) \quad (10)$$

This together with (3) leads to

$$E(k+1) = (I_N \otimes A - \mathcal{L}\otimes BK)E(k) + (\mathcal{L}\otimes BK)\Delta(k) - l(k)Q(((I_N \otimes A - \mathcal{L}\otimes BK)E(k) + (\mathcal{L}\otimes BK)\Delta(k))/l(k)) \quad (11)$$

Let $R(k) = \Delta(k)/l(k)$ and $S(k) = E(k)/l(k)$ .Next, we will prove that the quantizer is never saturated.

From Assumption 1, we can see that

$$\|(I_N \otimes A - \mathcal{L}\otimes BK)S(0) + (\mathcal{L}\otimes BK)R(0)\|_\infty \le \|A\|_\infty C_x / l_0 < L + 1/2 \quad (12)$$

Thus, the quantizer is unsaturated at time $k = 0$. In order to demonstrate the feasibility of the quantizer, mathematical induction is utilized. Assume that the quantizer is unsaturated for time $[0, k]$, i.e., $\sup_{1\le i\le k+1}\|S(i)\|_\infty \le 1/2r$.

Choose $\xi_i \in \mathbb{R}^N$ such that $\xi_i^T\mathcal{L} = \lambda_i\xi_i^T$. The unitary matrix $\Xi = [D1_N/\sqrt{N}, \xi_2,\cdots,\xi_N]$ is introduced such that $\Xi^T\mathcal{L}\Xi = diag(0,\lambda_2,\cdots,\lambda_N)$. It is easy to derive that

$$(\Xi\otimes I_n)^T(I_N \otimes A + \mathcal{L}\otimes BK)(\Xi\otimes I_n) = J(K) \quad (13)$$

Let $\tilde{R}(k) = (\Xi\otimes I_n)^T R(k)$ and $\tilde{S}(k) = (\Xi\otimes I_n)^T(\mathcal{L}\otimes BK)S(k)$, then

$$\tilde{R}(k+1) = r^{-1}J(K)\tilde{R}(k) - r^{-1}\tilde{S}(k) \quad (14)$$

Separate $\tilde{R}(k)$ into two parts $[\tilde{R}_1^T(k),\tilde{R}_2^T(k)]^T$, where $\tilde{R}_1(k) = \mathbf{0}_n$ represents the first n elements of $\tilde{R}(k)$. Similarly, $\tilde{S}(k)$ is in two parts $[\tilde{S}_1^T(k),\tilde{S}_2^T(k)]^T$. Then

$$\tilde{R}_2(k+1) = \left(\frac{J(K)}{r}\right)^{k+1}\tilde{R}_2(0) - \frac{1}{r}\sum_{i=0}^k\left(\frac{J(K)}{r}\right)^{k-i}\tilde{S}_2(i) \quad (15)$$

Combine Lemma 2 and the condition given in Theorem 1, we can get that $\|(J(K)/r)^k\| \le M\eta^k$, where $M = \sqrt{n(N-1)}(1+2/\varepsilon)^{n(N-1)-1}$ and $\eta = \rho(J(K)/r) + \varepsilon\|J(K)/r\| < 1$. This together with (15) leads to

$$\|\tilde{R}_2(k+1)\| \le M\eta^{k+1}\|\tilde{R}_2(0)\| + (1/r)M\sum_{i=0}^k \eta^{k-i}\|\tilde{S}_2(i)\| \quad (16)$$

And given $\|\tilde{R}_2(0)\| = \|R(0)\| \le 2\sqrt{Nn}C_x/l_0$, we have

$$(1/r)M\sum_{i=0}^k\eta^{k-i}\|\tilde{S}_2(i)\| \le n\sqrt{N}\lambda_N M\|BK\|_\infty(1-\eta^{k+1})/(2r^2(1-\eta)) \quad (17)$$

Since $\eta \in (0,1)$, (16) can be rewrote as

$$\|\tilde{R}_2(k+1)\| < \max\{\frac{2M\sqrt{Nn}C_x}{l_0}, \frac{n\sqrt{N}\lambda_N M\|BK\|_\infty}{2r^2(1-\eta)}\} \quad (18)$$

Noting that $l_0 > 4r^2C_x(1-\eta)/(\sqrt{n}\lambda_N\|BK\|_\infty)$, we have

$$\|(I_N \otimes A - \mathcal{L}\otimes BK)S(k+1) + (\mathcal{L}\otimes BK)R(k+1)\|_\infty < L + 1/2 \quad (19)$$

Thus, the $2L+1$ level uniform quantizer will not be saturated for time $k \ge 0$.

Considering $\|R(0)\|_\infty \le 2C_x/l_0$, then

$$\sup_{k\ge 0}\|R(k)\|_\infty < \max\{\frac{2C_x}{l_0}, \frac{n\sqrt{N}\lambda_N M\|BK\|_\infty}{2r^2(1-\eta)}\} < \infty \quad (20)$$

According to the definition of $R(k)$ and $0 < r < 1$, it implies that $\lim_{k\to\infty}\|\Delta(k)\|_\infty = 0$, so the multi-agent system (1) can realize the bipartite average consensus. □

Define the bipartite average consensus convergence rate as $r_{conv} = \sup_{\Delta(0)\ne 0}\lim_{k\to\infty}(\|\Delta(k)\|/\|\Delta(0)\|)^{1/k}$. From (16) and (17), for $\forall\Delta(0) \ne 0$, we get

$$\frac{\|\Delta(k+1)\|}{\|\Delta(0)\|} \le M(r\eta)^{k+1} + \frac{l_0 n\sqrt{N}\lambda_N M\|BK\|_\infty}{2r(1-\eta)\|\Delta(0)\|}r^k \quad (21)$$

Take the natural logarithm on both sides

$$\ln\left(\frac{\|\Delta(k+1)\|}{\|\Delta(0)\|}\right) \le \ln\left(\frac{l_0 n\sqrt{N}\lambda_N M\|BK\|_\infty}{2r(1-\eta)\|\Delta(0)\|}r^k\right) + \ln(1+o(1)) = k\ln r + O(1), k\to\infty \quad (22)$$

Thus,

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$$\lim_{k\to\infty}\left(\frac{\|\Delta(k+1)\|}{\|\Delta(0)\|}\right)^{1/(k+1)} \le \exp\left(\lim_{k\to\infty}\frac{1}{k+1}k\ln r + O(1)\right) = r \quad (23)$$

i.e., the bipartite average consensus convergence rate satisfies $r_{conv} \le r$.

3.2. *Structurally unbalanced graph*

The main result in this subsection is similar to Theorem 1. Please refer to theorem 1 for specific proof.

4. Conclusion

The consensus control problem of multi-agent systems with limited communication rate based on the signed connected undirected network topology is solved in this paper. For structurally balanced and unbalanced topology, the bipartite average consensus can be achieved by employing the proposed control scheme, respectively. In addition, the bipartite average consensus convergence rate is given explicitly.

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