# A Generalized Hamiltonian Conservative System with Multi-scroll Chaotic Flows

**Jingwen Liu*, Zhonggao Chen**
*College of Electronic Information and Automation, Tianjin University of Science and Technology,*
*Tianjin 300222，China*
*E-mail: * 1229778495@qq.com*
*www.tust.edu.cn*

**Abstract**

By analyzing mechanics and energy of a three-dimensional volume conservative chaotic system proposed by Vaidyanathan and Volos, a new generalized conservative chaotic system with multi-scroll chaotic flows is found based on the corresponding Hamiltonian energy. The new system satisfies both volume conservation and energy conservation. By analyzing the equilibrium characteristics of the system, equilibrium points of the new system are found to be a line. In addition, the number of scrolls of conservative chaotic flows of the new system depend on the corresponding Hamiltonian energy. The paper provides a new conservative chaotic model for chaos application.

*Keywords*: conservative, multi-scroll, Hamiltonian energy, equilibrium

## 1. Introduction

Chaos is a special physical phenomenon in nonlinear systems[1], which widely exists in the fields of physics and life science. Therefore, people are committed to constructing chaotic systems with better performance[2]. Compared with single scroll chaotic systems or double scroll chaotic systems, multi-scroll chaotic systems have more control parameters and more corresponding key parameters. Moreover, it can present complex multi-direction grid scroll in phase space. The number and shape of scroll can also be controlled and adjusted by the parameters of the system[3]. In practical application, the increase of scrolls number in chaotic systems are realized by increasing the number of equilibrium points. Multi-scroll chaotic attractors can show more complex chaotic dynamic behaviors[4-6], and has high application value in chaotic information processing[10], chaotic neural network[7] and chaotic secure communication[12], which makes the multi scroll chaotic system have a very broad application prospect in practical engineering[10-13]. Therefore, the research of multi scroll chaotic system is becoming a research hotspot in the field of chaos[14-18].

However, there are few studies on the multi-scroll flows of conservative chaotic systems. In this paper, single direction multi-scroll flows and double direction multi-scroll flows are constructed and their direction and number are both controllable. At the same time, the feasibility and effectiveness of the method of constructing multi-scroll flows is verified by Lyapunov exponent spectrum and phase diagram, which provides a new method for the construction of multi-scroll flows and a new conservative chaotic model for chaos application.

## 2. Construction of Four-dimension Conservative Chaotic System

Firstly, a new four-dimension conservative system is found by analyzing mechanics and energy of a three-dimensional volume conservative chaotic system proposed by Vaidyanathan and Volos[19]. It can be expressed as

$$\dot{\mathbf{x}} = J(\mathbf{x})\nabla H(\mathbf{x}). \quad (1)$$

Where, $J(\mathbf{x})=\begin{bmatrix}0 & a & x & 0\\ -a & 0 & y & 0\\ x & -y & 0 & 1+2w\\ 0 & 0 & -1-2w & 0\end{bmatrix}$, $\nabla H(\mathbf{x})=[x\ y\ z\ 1]^T$.

Secondly, Equation (1) can be described as

$$\begin{cases}\dot{x}=ay+xz\\ \dot{y}=-ax+yz\\ \dot{z}=1-x^2-y^2-2w\\ \dot{w}=-z-2zw\end{cases}. \tag{2}$$

Where $x, y, z$ and $w$ are state variable, and $a$ is a system parameter, the divergence of the system (2) is

$$\nabla f=\frac{\partial \dot{x}}{\partial x}+\frac{\partial \dot{y}}{\partial y}+\frac{\partial \dot{z}}{\partial z}+\frac{\partial \dot{w}}{\partial w}=0. \tag{3}$$

Moreover, the derivative of Hamiltonian energy is

$$\dot{H}=\nabla H(\mathbf{x})^T J(\mathbf{x})\nabla \mathrm{H}(\mathbf{x})=0. \tag{4}$$

According to the above analysis, the system satisfies both Hamiltonian energy conservation and volume conservation. Meanwhile, the equilibrium equation of system (2) is

$$\begin{cases}ay+xz=0\\ -ax+yz=0\\ 1-x^2-y^2-2w=0\\ -z-2zw=0\end{cases}. \tag{5}$$

It can be found that the equilibrium point of the system (2) is obtained as $(0,0,z,-0.5)$, which are found to be a line.

## 3. Construction of Conservative Chaotic Systems with Multiple multi-scroll flows

In this part, multi-scroll flows are obtained by changing Hamiltonian energy. According to Equation (1), the Hamiltonian energy of system (2) is obtained as

$$H(\mathbf{x})=\frac{1}{2}(x^2+y^2+z^2)+w. \tag{6}$$

Set system parameter $a=1$, the initial value $(x,y,z,w)=(x(0),1,1,1)$, Lyapunov exponent spectrum of th e system (2) is shown in Fig. 1.

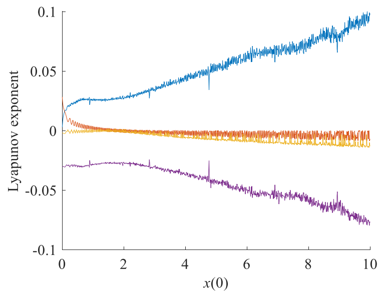



Fig. 1. Lyapunov exponent spectrum of system (2)

Set the initial value $(x,y,z,w)=(\frac{\pi}{2},1,1,1)$, Single scroll flow is shown in Fig. 2.

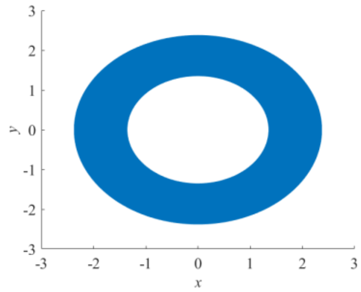



Fig. 2. Single scroll flow

### 3.1. Characteristic analysis of the single direction multi-scroll conservative system

In order to obtain the multi-scroll conservative system, the Hamiltonian energy is firstly changed by introducing piecewise functions. Set $\nabla H(\mathbf{x})=[f(x)\ y\ z\ 1]^T$, and Equation (2) can be described as

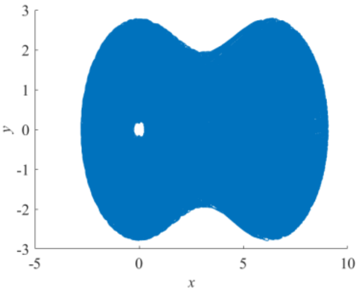
$$\begin{cases}\dot{x}=ay+xz\\ \dot{y}=-af(x)+yz\\ \dot{z}=1-xf(x)-y^2-2w\\ \dot{w}=-z-2zw\end{cases}, \tag{7}$$

$$f(x)=\begin{cases}x+Q, & x<-Q\\ sin(x), & -N\le x\le Q\\ x-N & x>N\end{cases}. \tag{8}$$

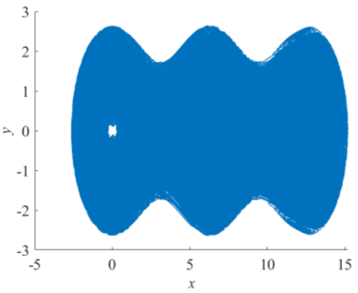
Where $f(x)$ satisfy mapping, and $Q=n_1\pi, N=n_2\pi, n_1, n_2\in Z^*$.The Hamiltonian energy of system (2) is obtained as

$$H(\mathbf{x})=\frac{1}{2}(y^2+z^2)+w+\int f(x)\,dx. \tag{9}$$

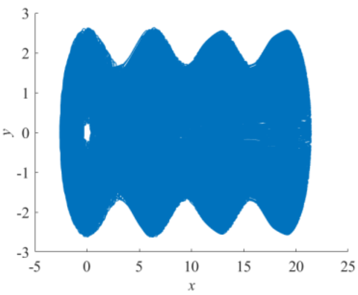
To further explain the process of generating multi-scroll in system (7), set $Q=0$ and the initial value $(x,y,z,w)=(\frac{\pi}{2},1,1,1)$. When $N=2\pi, 4\pi$ and $6\pi$, a 2-scroll flow, a 3-scroll flow and a 4-scroll flow extended in the positive direction of x-axis are respectively shown in Fig. 3.



(a) 2-scroll flow



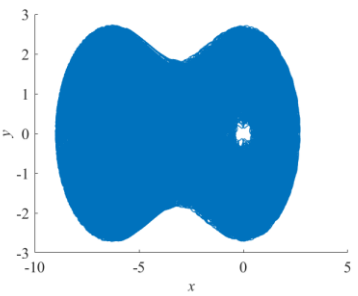
(b) 3-scroll flow



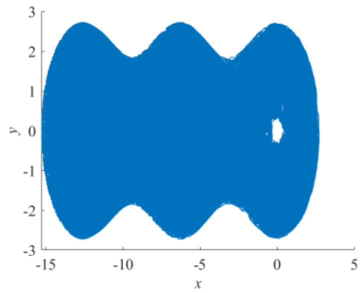
(c) 4-scroll flow

Fig. 3. Multi-scroll flows of the system (7)

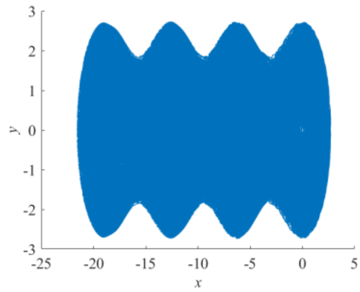
Set $N = 0$ and the initial value $(x, y, z, w) = (\frac{\pi}{2}, 1, 1, 1)$.When $Q = 2\pi, 4\pi$ and $6\pi$, a 2-scroll flow, a 3-scroll flow and a 4-scroll flow extended in the negative direction of x-axis are respectively shown in Fig. 4.



(a) 2-scroll flow



(b) 3-scroll flow



(c) 4-scroll flow

Fig. 4. Multi-scroll flows of the system (7)

In summary, it can be concluded that flows with different numbers of scrolls can be obtained by changing the values of N and Q to change the Hamiltonian energy of the system (7), and both the direction and the number of scrolls is controllable.

### *3.2. Characteristic analysis of the double direction multi-scroll conservative system*

In the above, multi-scroll flows along the x-axis direction are obtained in system (7) by changing the Hamiltonian energy of x variable. Similarly, grid type multi-scroll flows along x and y axis directions can be simultaneously obtained by changing the Hamiltonian energy of x variable and y variable. Set $\nabla H(\mathbf{x}) = [f(x)\, f(y)\, z\, 1]^T$, and Equation (2) can be described as

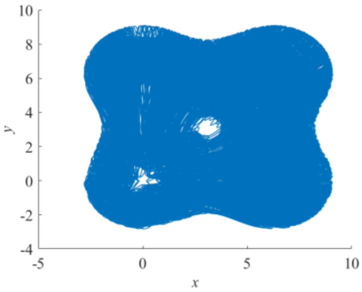
$$\begin{cases} \dot{x} = af(y) + xz \\ \dot{y} = -af(x) + yz \\ \dot{z} = 1 - xf(x) - yf(y) - 2w' \\ \dot{w} = -z - 2zw \end{cases} \tag{10}$$

$$f(x) = \begin{cases} x + Q_1, & x < -Q_1 \\ sin(x), -N_1 \le x \le Q_1, \\ x - N_1, & x > N_1 \end{cases} \tag{11}$$

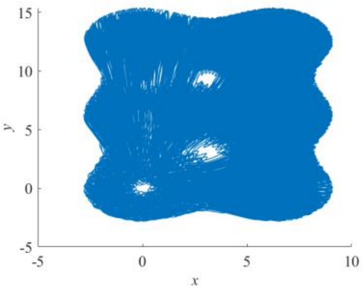
$$f(y) = \begin{cases} y + Q_2, & y < -Q_2 \\ sin(y), -N_2 \le y \le Q_2, \\ y - N_2, & y > N_2 \end{cases} \tag{12}$$

Where $f(x)$ and $f(y)$ respectively satisfy mapping (11) and mapping (12), and $Q_1 = m_1\pi, Q_2 = m_2\pi, N_1 = m_3\pi, N_2 = m_4\pi, m_1, m_2, m_3, m_4 \in Z^*$. The Hamiltonian energy of system (10) is obtained as

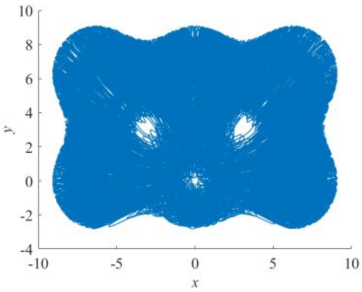
$$H(\mathbf{x}) = \frac{1}{2}z^2 + w + \int f(y)\, dy + \int f(x)\, dx. \quad (13)$$



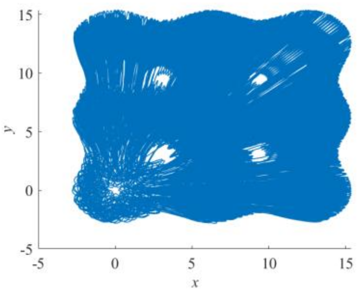

(a) $2 \times 2$ grid type multi-scroll flow




(b) $2 \times 3$ grid type multi-scroll flow




(c) $3 \times 2$ grid type multi-scroll flow




(d) $3 \times 3$ grid type multi-scroll flow

Fig. 5. Grid type multi-scroll flows of the system (10)

To further explain the process of generating multi-scroll in system (10), set the initial value $(x, y, z, w) = (\frac{\pi}{2}, \frac{\pi}{2}, 1, 1)$. When $Q_1 = Q_2 = 0, N_1 = N_2 = 2\pi$, a $2 \times 2$ grid type multi-scroll flow is shown in Fig. 5 (a). When $Q_1 = Q_2 = 0, N_1 = 2\pi, N_2 = 4\pi$, a $2 \times 3$ grid type multi-scroll flow is shown in Fig. 5 (b). When $Q_2 = Q_1 = 0, N_1 = 4\pi, N_2 = 2\pi$, a $3 \times 2$ grid type multi-scroll flow is shown in Fig. 5 (c). When $Q_1 = Q_2 = 0, N_1 = N_2 = 4\pi$, a $3 \times 3$ grid type multi-scroll flow is shown in Fig. 5 (d).

In summary, it can be concluded that different grid type multi-scroll flows can be obtained by changing the values of $Q_1, Q_2, N_1$ $and$ $N_2$ to change the Hamiltonian energy of the system (10),, and both the direction and the number of scrolls is controllable.

## 4. Conclusion

In this paper, a new generalized conservative chaotic system with multi-scroll chaotic flows is found based on the corresponding Hamiltonian energy through the mechanical and energy analysis of the three-dimensional volumetric conservative chaotic system proposed by Vaidyanathan and Volos. The new system satisfies the conservation of volume and energy. At the same time, the feasibility and effectiveness of the method of constructing multi-scroll flows is verified by Lyapunov exponent spectrum and phase diagram, which provides a new method for the construction of multi-scroll flows and a new conservative chaotic model for chaos application.

## References


1. Norbert Euler,Maria Clara Nucci & Da Jun Zhang(2021).*Nonlinear Systems and Their Remarkable Mathematical Structures, Volumes 1, 2, and 3*.CRC Press
2. Xiu Zhao, et al."Complex generalized synchronization of complex-variable chaotic systems." *The European Physical Journal Special Topics* .prepublish(2021): doi:10.1140/EPJS/S11734-021-00129-6.
3. Complexity Research; New Complexity Research Study Findings Reported from Civil Aviation University of China (A New Memristor-based 5d Chaotic System and Circuit Implementation. *Technology News Focus* .(2019): doi:
4. Chen,D.Y;Sun,Z.T.;Ma,X.Y.;Chen,L.Circuit implement ation and model of a new multi-scroll chaotic system.Int.J.CircuitTheoryAppl.2014,42,407–424.
5. Deng, W.H.; Lü, J.H. Generating multi-directional multi-scroll chaotic attractors via a fractional differential hysteresis system. Phys. Lett. A 2007, 369, 438–443.

6. Zhang, Y .; Yu, S.M.; Liu, M.H. Generating multi-scroll hyperchaotic attractors based on FPGA technology. Circuits Syst. 2007, 12, 39–43.
7. Samaneh Gholami,etal."An efficient image-based verification scheme by fusion of double random phase encoding and dynamic chaotic map." Multimedia Tools and Applications 78.17(2019),10.1007/s11042-019-7714-1.
8. Yongfeng Cui,,Zhongyuan Zhao,,Yuankun Ma & Shi Dong.Resource allocation algorithm design of high quality of service based on chaotic neural network in wireless communication technology. Cluster Computing(5),2019,10.1007/s10586-017-1285-6.
9. Lili Zhou & Fei Tan.A chaotic secure communication scheme based on synchronization of double-layered and multiple complex networks. Nonlinear Dynamics(2), 2019,10.1007/s11071-019-04828-7.
10. Yu, S.M.; Lü, J.H.; Leung, H.; Chen, G.R. Design and implementation of n-scroll chaotic attractors from a general jerk circuit. IEEE T rans. Circuits Syst. I Regul. Pap. 2005, 52, 1459–1476.
11. He, S.B.; Sun, K.H.; Wang, H.H.; Ai, X.X.; Xu, Y .X. Design of n-dimensional multi-scroll Jerk chaotic system and its performance. J. Appl. Anal. Comput. 2016, 6, 1180–1194.
12. Liu, X.Z.; Shen, X.M.; Zhang, H.T. Multi-scroll chaotic and hyperchaotic attractors generated from Chen system. Int. J. Bifurc. Chaos 2012, 22, 1250033.
13. Chen, Z.; Wen, G.L.; Zhou, H.A.; Chen, J.Y . A new M x N-grid double-scroll chaotic attractors from Rucklidge chaotic system. Optik 2017, 136, 27–35.
14. Tlelo-Cuautle, E.; Rangel-Magdaleno, J.J.; Pano-Azucena, A.D.; Obeso-Rodelo, P .J.; Nunez-Perez, J.C. FPGA realization of multi-scroll chaotic oscillators. Commun. Nonlinear Sci. Numer. Simul. 2015, 27, 66–80.
15. Luo, X.H. Circuitry implementation of a novel nonautonomous hyperchaotic Liu system based on sine input. Chin. Phys. B 2009, 18, 3304–3308.
16. Hu, X.; Liu, C.; Liu, L.; Ni, J.K.; Li, S.L. Multi-scroll hidden attractors in improved Sprott A system. Nonlinear Dyn. 2016, 86, 1725–1734.
17. Tang, W.K.S.; Zhong, G.Q.; Chen, G.; Man, K.F. Generation of n-scroll attractors via sine function. IEEE T rans. Circuits Syst. I Fundam. Theory Appl. 2001, 48, 1369–1372.
18. Chen Z G, Song M H. Dynamic Characteristics Analysis of a Multi-scroll Conservative Chaotic System with Sinusoidal Nonlinearity. The 2021 International Conference on Artificial Life and Robotics (ICAROB2021), Jan. 21-24.
19. Jia H Y, Chen Z G, Shi W X, et al. A generalized Hamiltonian conservative chaotic system with multi-stable flows. Journal of Shandong University (Engineering Science), 2021, 51 (6).


**Authors Introduction**


Ms. Jingwen Liu

She is studying for a master's degree in electronic information at Tianjin University of Science and Technology.

Mr. Zhonggao Chen

He received the B.S. degree from Tianjin University of Science and Technology, Tianjin, China. And now she is a master course student. Her main research interests are research and circuit implementation of multi-scroll Hamiltonian energy conservative system.