Mapping the Motion of Highly-inclined Triple System into a Secular Perturbation Model

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Abstract

The motions of three bodies like Sun-Asteroid-Jupiter system or triple star system are formalized as hierarchical three body problem. When the third body orbits around the rest in a highly inclined elliptic orbit, the system undergoes the oscillation, called the Kozai oscillation, where the eccentricity may increase with decrease of the inclination of the orbital plane.

While the Kozai oscillation seems to be a key process in orbital evolution, including disruption of triple system, its reflection into actual trajectories is quite complicated to analyze. For this reason, we try to map these trajectories into a secular perturbation model with data assimilation and demonstrate the extraction of state and its transition (libration to circulation and vice versa) as the Kozai oscillation.

Keywords: List four to six keywords which characterize the article.

1. Introduction

The motion of gravitationally interacting three bodies is called the three-body problem and known to be analytically unsolvable unlike to the two-body problem, in which the bodies draw elliptic orbits. There are two contrastive types in motion of three bodies. In one type, the bodies interact each other in an extremely complicated way and often end in the disruption into a pair of bodies and the rest one. This type of motion is generally called chaotic. The other type of motion is that bodies draw hierarchical elliptic orbits, each of which is similar to that of two-body problem, with gradual change in its shape. The latter may be realized when there is an enough contrast in the masses (e.g. the Sun occupies 99.8% of the entire solar system's mass) or in orbital radii of the inner and outer orbits.

Our interest is in between the two. If two elliptic orbits (a schematic illustration shown in see Fig. 1) are initially placed close to each other, the outer orbit evolve to a hyperbolic orbit and the third-body escapes on it. How close the system can be allocated without such a disruption of the system is called stability limits and has been studied since Harrington[1]. While the contribution of Kozai mechanism, explained later, to the instability of hierarchical triple systems has been pointed out[2], the process until the system are finally broken is still unclear. Partially, the difficulty comes from a necessity of a long numerical orbit to see the sign of instability as well as a complicated process being involved till the disintegration of the system. In this study, we construct a mapping from the motion of three bodies in a Cartesian flame into orbital elements by utilizing the most simple secular perturbation model as auxiliary dynamical system.

2. Method

2.1. *Equation of Motion*

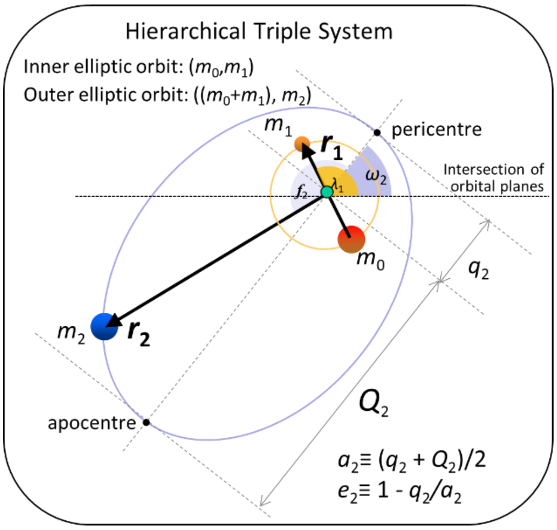
Let m_0, m_1 and m_2 be the masses of bodies gravitationally interacting three bodies. We introduce Jacobian coordinates to describe their motion, that is, $\boldsymbol{r}_1$ is the vector from m_0 to m_1, and $\boldsymbol{r}_2$ from their barycenter to m_2. Equations of motion of these bodies are given by

$$\mu_i^* \frac{d^2 \boldsymbol{r}_i}{dt} = -\frac{G\mu_i^* \sum_{j=0}^{i} m_j}{||\boldsymbol{r}||^2} + \frac{\partial R}{\partial \boldsymbol{r}_i} \quad (1)$$

$(i = 1,2)$ with disturbing function

$$R = \frac{Gm_0 m_1}{||\boldsymbol{r}_{01}||} + \frac{Gm_2 m_0}{||\boldsymbol{r}_{02}||} - \frac{Gm_2(m_0+m_1)}{||\boldsymbol{r}_2||},$$

where vectors $\boldsymbol{r}_{ij}$ from m_0 to m_1 and reduced masses $\mu_1^* = \frac{m_0 m_1}{m_0+m_1}, \mu_2^* = \frac{m_2(m_0+m_1)}{m_0+m_1+m_2}$.



2.2. *Secular Perturbation and Kozai Oscillation*

A solution $\boldsymbol{r}_1(t)$ and $\boldsymbol{r}_2(t)$ of the equations of motion Eq. (1) defined above generally draw nearly elliptic orbits in short-term, changing gradually their shape change in long-term. When our interest is long-term evolution of the system, it is good to rewrite the equations of motion with respect to variables describing the orbital shapes, called orbital elements, under a certain approximation neglecting short-term variation.

Orbital elements consist of $a_i, e_i, q_i := a_i(1 - e_i), \omega_i$, corresponding to $\boldsymbol{r}_i$ and its derivative, as well as angle I between the orbital planes on which $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are (see Fig.1 for geometrical definitions). With these variables, the disturbing function is rewritten as

$$R = \frac{Gm_2\alpha^2}{16a_2}[(3\cos^2 I - 1)(2 + 3e_1^2 + 3e_2^2) + 15e_1^2 \sin^2 I \cos 2\omega_1], \quad (2)$$

where $\alpha := a_1/a_2$ and e_i are kept up to their second order. Equation (1) is accordingly transformed to the first order differential equations w.r.t. these orbital elements, called planetary equations (Note that we have derived Eq.(2) using the algorithm[5] for computer algebra aiming at a higher order expansion for future study. For this reason, its exact form is slightly different than its traditional form[4]).

The equations derived by Eq. (2) has only solutions such that ω_1, indicating the direction of the pericenter measured from the intersection of two orbital planes, circulates (i.e. ω_1 takes all possible values from $0°$ to $360°$) if the inclination angle I of one plane against the other, is small, whereas another type of solutions exists for sufficiently high I, where ω_1 oscillates a limited range including $+90°$ or $-90°$. The former type of motion is called the circulation, and the latter the libration. When I is high so that the libration is possible, e_1 significantly rise up or down along with the entire period of ω_1, as well as I varies anti-correlatedly to e_1. This oscillation of e_1 and I is called Kozai mechanism[4], which is originally studied by Yoshihide Kozai for the cases of $m_1 \to 0$.

2.3. *Introduction of Stochastic change*

While we aim at extract such a process into the disintegration as a variation of orbital elements and consider mapping the outcome of the full model Eq. (1) to the perturbation model Eq. (2), the discrepancy is not small between them. For this reason, we introduce a stochastic process and allow the solutions of Eq. (2) to jump at each times step by adding the realization of random variables. Specifically, an extended system

$$(e_{1,n}, \omega_{1,n}) = \mathrm{RK4}(e_{1,n-1} + \delta e_{1,n}, \omega_{1,n} + \delta\omega_{1,n}) \quad (3)$$
$$\delta e_{1,n} \sim N(0, \sigma_e^2), \delta\omega_{1,n} \sim N(0, \sigma_\omega^2),$$

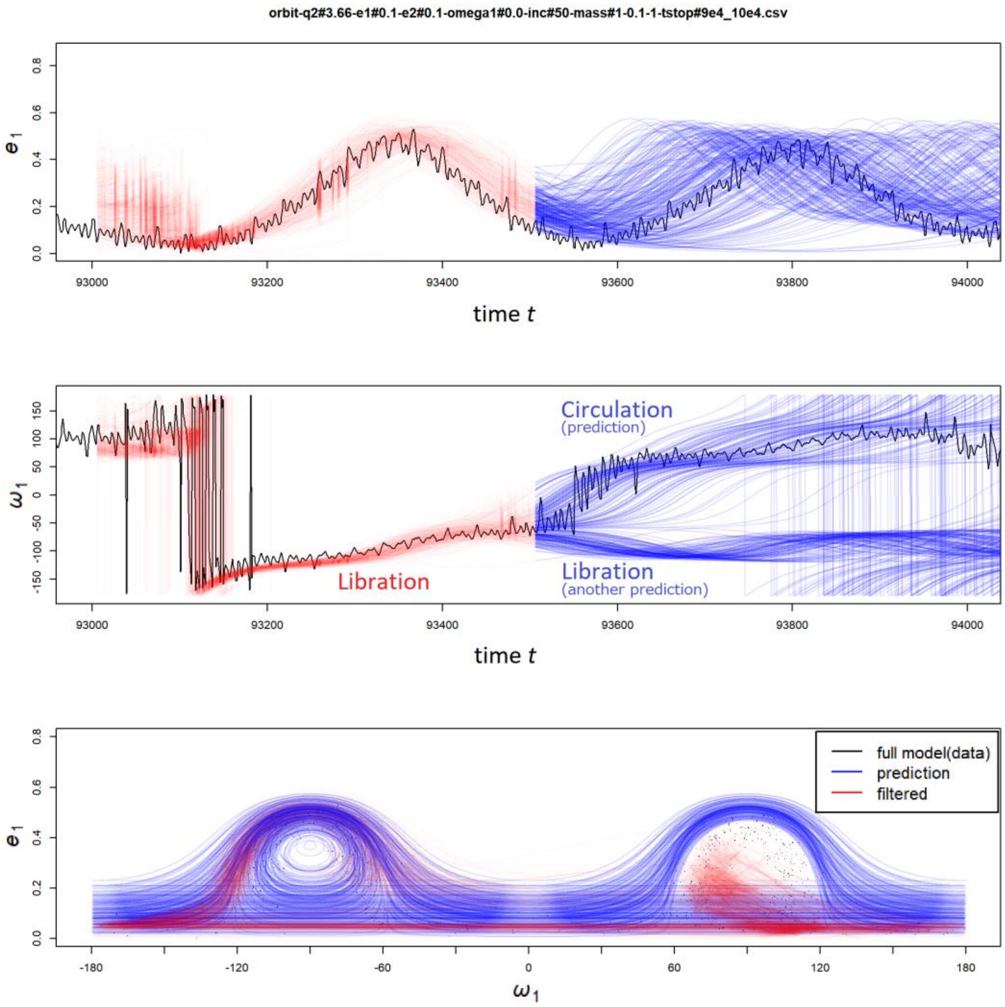


Fig. 2. Mapping of a solution of equations of motion to secular perturbation model. The solution of Eq (1) is shown in black (full model), corresponding mapped trajectories to secular perturbation model using particle filtering in red (filtered), and propagation from the last filtered time t = 93,500 in blue (prediction). Parameters are $m_1 = 0.1, m_0 = m_2 = 1,\ q_2/a_1 = 3.66,\ e_1 = e_2 = 0.1, I = 50°, \omega_1 = 0°, \omega_2 = 90°$.

is used instead of Eq.(2), where RK4(e_1, ω_1) is a propagator which advances the time by a given amount Δt, following to the planetary equations generated from Eq. (2), with parameters $\Delta t = 1, \sigma_e = \sigma_\omega/2\pi = 0.002$.

The discrepancy in (e_1, ω_1) between Eqs. (1) and(3) are measured by the likelihood for a single time point based on a normal,

$$\ln p\left(e_{1,n}^{\rm obs}, \omega_{1,n}^{\rm obs} | e_{1,n}, \omega_{1,n}\right) = \left(e_{1,n}^{\rm obs} - e_{1,n}\right)^2 + \left(\omega_{1,n}^{\rm obs} - \omega_{1,n}\right)^2/\pi^2 . \tag{4}$$

Here we regard the outcome of Eq. (1) as observation data (denoted by $e_{1,n}^{\rm obs}, \omega_{1,n}^{\rm obs}$), and that of Eq. (3) as the latent variables $(e_{1,n}, \omega_{1,n})$. Eqs. (3) and (4) form the state space model, to which sequential Bayesian estimation algorithms[6,7] are applicable. Of these algorithms, we implement the mapping from the “data” and the “latent” using Particle Filtering[6,7].

3. Results

We will demonstrate how an exact solution of the equations of motion Eq. (1) is mapped to the secular perturbation model, and that a transition between the libration and the circulation are identified.

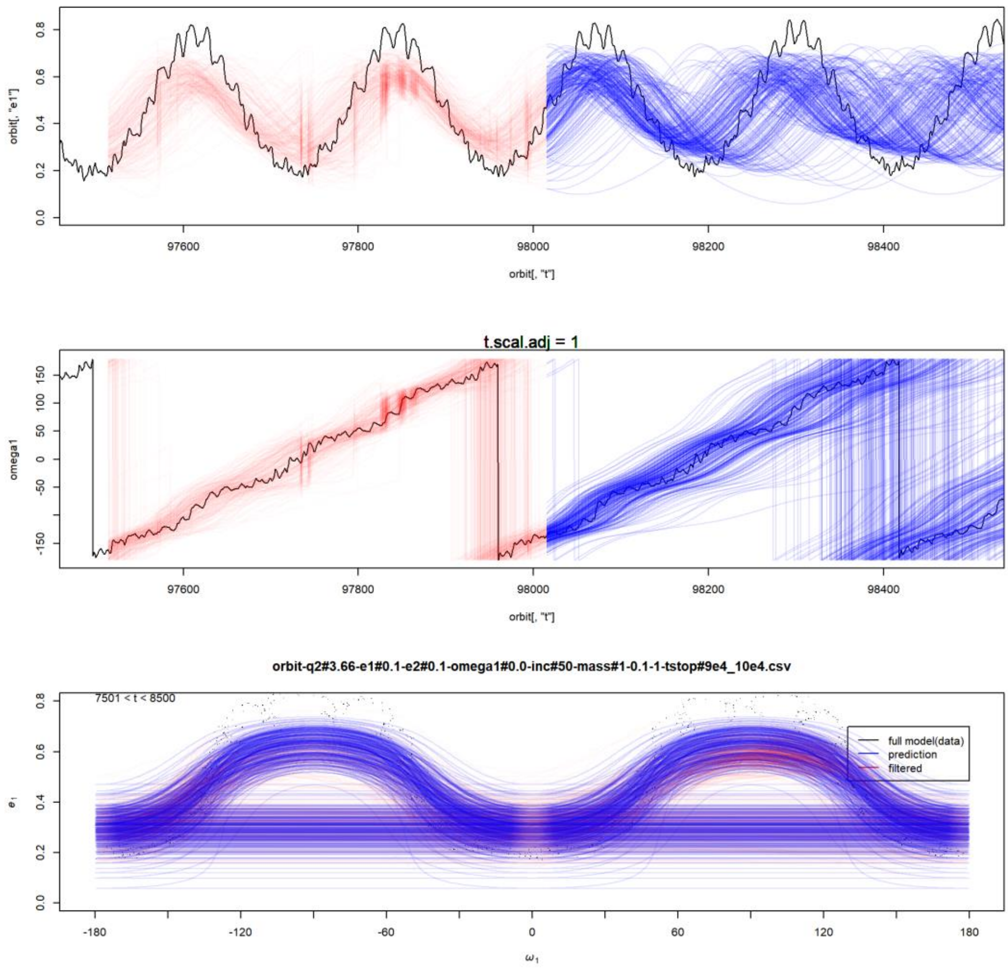


Fig. 3. The same orbit as in Fig. 2, but a later time range being covered.

An example shown in Fig. 2 includes a transition from the libration the circulation, followed by an elevation of e_1 in Fig. 3. Before going to a detailed inspection of these figures, we remark on the choice of the configuration. Masses and initial orbital parameters are chosen as $m_1 = 0.1$, $m_0 = m_2 = 1$, orbital separation $q_2/a_1 = 3.66$, eccentricities $e_1 = e_2 = 0.1$, and mutual inclination $I = 50°$ (the entire parameters are shown in the caption) so that the system undergoes the 1:6 mean motion resonance (MMR). MMRs may cause orbital instability and in fact we have observed a disruption of the system at the 1:6 MMR under more massive m_1 (specifically $m_1 = 1$). The process to the disruption is as follows: e_1 suddenly increases before increase of e_1 with e_2 after its long lasting quasi periodic variation, the increase of e_1 forces the increase of e_2, by which a_2 increases to go a hyperbolic orbit. Interestingly, the increase of e_1 often occurs when the system in a libration around $\omega_1 = \pm 90°$. This is the reason why we consider Kozai mechanism enhances the instability first realized by MMR. While the initial condition for Fig. 2 is chosen Kozai mechanism and a MMR coexists, we restricted ourselves rather less massive $m_1 = 0.1$ because the setting of $m_1 = 0.1$ provides too strong perturbation to adequately describe the motion with our simple perturbation model. Though under this lowered mass the system disruption does not occur, an elevation of e_1 is expected, which is a necessary step to the system disintegration.

First we confirm that the transition to the circulation is captured via mapping. In Fig. 2, a quasi-periodic sustains for t < 94,000, then range of e_1 gradually increases to reach 0.8 at maximum. There is a transition from libration around $\omega_1 = -90°$, when e_1 tends to raise up. We use particle filtering to learn the time course of (e_1, ω_1) provided by the solution of Eq. (1). The trajectories of respective particles forming filtered distribution are shown in red, followed by trajectories propagated from the last filtered state (t = 93,500) without stochastic jumps (i.e. just solving planetary equations), shown in blue. Time t = 93,500 is close to the transition to circulation. Some propagated trajectories keep in the libration, while others transit to circulation. Hence, we can say that our perturbation model with stochastic jumps can capture the transition between the two states. The ratio of the numbers of orbits in the two mode may be interpreted an probabilistic evaluation of which state the orbit is in, when a large number of orbits need to be classified in a systematic way.

Continuing the filtering and the prediction after the circulation, we see that the perturbation model catches up the increase of e_1 and shorted period of the Kozai oscillation. However, the maximum value of e_1 given by the model is that out full simulation outcome, which reaches to 0.8. Higher order terms neglected here may be necessary to improve the agreement.

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Authors Introduction

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