The Eighteenth International Symposium on Artificial Life and Robotics 2013 (AROB 18th '13),
Daejeon Convention Center, Daejeon, Korea, January 30-February 1, 2013

# Self-Triggered Optimal Control Based on Optimization with Prediction Horizon One

Koichi Kobayashi[1] and Kunihiko Hiraishi[1]

[1]Japan Advanced Institute of Science and Technology, Japan
(Tel: +81-761-51-1282; E-mail: k-kobaya@jaist.ac.jp)

**Abstract:** Self-triggered control is a control method that the control input and the sampling period are computed simultaneously in sampled-data control systems, and is extensively studied in the field of networked control systems. In this paper, a new approach for self-triggered control is proposed based on model predictive control. First, the optimal control problem with horizon one, in which the first sampling period and the control input are found, is formulated. By solving this problem at each sampling interval, self-triggered model predictive control is realized. Next, an iterative solution method is proposed. In this solution method, a quadratic programming problem is repeatedly solved. Finally, the effectiveness of the proposed approach is shown by a numerical example.

**Keywords:** optimal control, networked control systems, self-triggered control.

## 1. INTRODUCTION

In recent years, analysis and synthesis of networked control systems (NCSs) have been extensively studied [1, 4]. An NCS is a control system in which plants, sensors, controllers, and actuators are connected through communication networks. In distributed control systems, subsystems are frequently connected via communication networks, and it is important to consider analysis and synthesis of distributed control systems from the viewpoint of NCSs. In design of NCSs, several technical issues such as packet losses and transmission delays are included. However, it is difficult to consider these issues in a unified way, and it is suitable to discuss an individual problem. From this viewpoint, several results have been obtained so far (see, e.g., [6-9]).

In this paper, the periodic paradigm is focused as one of the technical issues in NCSs. The periodic paradigm is that the controller is periodically executed at a given unit of time. The period is chosen based on CPU processing time, communication bandwidth, and so on. However, in NCSs, communication should occur, when there exists important information, which must be transmitted from the controller to the actuator and/or from the sensor to the controller. In this sense, the periodic paradigm is not necessarily suitable, and it is important to consider a new approach for design of NCSs. As one of the methods to overcome this drawback of the periodic paradigm, self-triggered control has been proposed so far (see, e.g., [2, 3, 5, 10, 13]). In self-triggered control, the next sampling time at which the control input is recomputed is computed. That is, both the sampling period and the control input are computed simultaneously. In many existing works, first, the continuous-time controller is obtained, and after that, the sampling period such that stability is preserved is computed. However, few results on optimal control have been obtained so far. From the viewpoint of optimal control, for example, a design method based one-step finite horizon boundary has been recently proposed in [12, 14]. In this method, the first sampling period such that the optimal value of the cost function is improved, is computed under the constraint that other sampling periods are given as a constant. However, a nonlinear equation must be solved. Furthermore, input constraints cannot be considered in this method.

In this paper, a new approach for self-triggered control is proposed based on model predictive control (MPC). First, the optimal control problem with horizon one, in which the first sampling period and the control input are found, is formulated. By solving this problem at each sampling interval, self-triggered model predictive control is realized. Next, an iterative solution method is proposed. In this solution method, quadratic programming (QP) problems are repeatedly solved. Then, we can impose input constraints for the system. Finally, the effectiveness of the proposed approach is shown by a numerical example. The proposed approach provides us a basic result for self-triggered optimal control.

**Notation:** Let $\mathcal{R}$ denote the set of real numbers. Let $I_n, 0_{m\times n}$ denote the $n\times n$ identity matrix, the $m\times n$ zero matrix, respectively. For simplicity, we sometimes use the symbol $0$ instead of $0_{m\times n}$, and the symbol $I$ instead of $I_n$.

## 2. PROBLEM FORMULATION

Consider the following continuous-time linear system:

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x \in \mathcal{R}^n$ is the state, and $u \in [u_{\min}, u_{\max}] \subseteq \mathcal{R}^m$ is the control input. $u_{\min}, u_{\max} \in \mathcal{R}^m$ are given constants, and the interval $[u_{\min}, u_{\max}]$ expresses input constraints. By $t_k$, $k = 0, 1, \dots$, denote the sampling time, and by $h_k := t_{k+1} - t_k$, denote the sampling period. Assume that the control input is piecewise constant, that is, the control input is given as

$$u(t) = u(t_k),\ \ t \in [t_k, t_{k+1}).$$

Hereafter, we denote $u(t_k)$ as $u_k$. In addition, assume that a pair $(A, B)$ is controllable.

Before the problem studied here is formulated, some preparations are given. Suppose that $h_0$ is a decision variable, and $h_i = h$, $i \geq 1$ is satisfied, where $h \geq 0$ is a given constant. We also suppose that the input constraint is imposed in the time interval $[t_0, h_0+h(N-1))$, where $N \geq 1$ is a given integer. Note here that the input constraint is not imposed in the time interval $[h_0 + h(N-1), \infty)$. Then, consider the following cost function

$$J = J_1 + J_2 + J_3, \tag{2}$$

$$J_1 = \int_{t_0}^{h_0} \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt,$$

$$J_2 = \int_{h_0}^{h_0+h(N-1)} \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt,$$

$$J_3 = \int_{h_0+h(N-1)}^{\infty} \left\{ x^T(t)Qx(t) + u^T(t)Ru(t) \right\} dt = x^T(h_0+h(N-1))P(h)x(h_0+h(N-1)).$$

In the above cost function, $P(h)$ is a symmetric positive definite matrix, which is a solution of the following discrete-time algebraic Riccati equation

$$\tilde{A}^T(h)P(h)\tilde{A}(h) - P(h) - (\tilde{A}^T(h)P(h)\tilde{B}(h) + \tilde{S}(h)) \times(\tilde{B}^T(h)P(h)\tilde{B}(h) + \tilde{R}(h))^{-1} \times(\tilde{B}^T(h)P(h)\tilde{A}(h) + \tilde{S}^T(h)) + \tilde{Q}(h) = 0$$

where

$$\tilde{A}(h) := e^{Ah}, \ \ \tilde{B}(h) := \int_0^h e^{At}dtB$$

and

$$\begin{bmatrix} \tilde{Q}(h) & \tilde{S}(h) \\ \tilde{S}^T(h) & \tilde{R}(h) \end{bmatrix} := \int_0^h e^{F^Tt} \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix} e^{Ft}dt,$$

$$F := \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}.$$

Since the input constraint is not imposed in the time interval $[h_0 + h(N-1), \infty)$, the optimal value of $J_3$ can be explicitly characterized by $x(h_0 + h(N-1))$.

Under the above preparation, consider the following problem.

**Problem 1:** Suppose that for the system (1), the initial time $t_0$, the initial state $x(0) = x_0$, $h_1 = h_2 = \cdots = h$, $N \geq 1$, and $\gamma \geq 1$ are given. Then find a control input sequence $u_0, u_1, \ldots, u_{N-1}$ maximizing a sampling period $h_0$ under the following constraints

$$h \leq h_0 \leq h_{\max},$$

$$u_{\min} \leq u_k \leq u_{\max}, \ \ k = 0, 1, \ldots, N-1, \tag{3}$$

$$J^*_{h_0} \leq \gamma J^*_h \tag{4}$$

where $h_{\max} \geq 0$ is a given constant, and $J^*_{h_0}$ is the optimal value of the cost function of (2), $J^*_h$ is the optimal value of the cost function of (2) under $h_0 = h$. ■

In Problem 1, control performance can be adjusted by suitably giving $\gamma$. We remark that in this problem, $h_0$ is maximized under some constraints. Furthermore, in this problem, a control input sequence is computed, but on sampling periods, only the first one ($h_0$) is computed. In this sense, this problem is regarded as a kind of the optimal control problem with prediction horizon one.

# 3. PROPOSED SOLUTION METHOD

First, for a fixed $h_0$, consider deriving $J^*_{h_0}$. The value of $J^*_h$ can be derived by a similar method. The value of $J^*_{h_0}$ is given as the optimal value of the following optimal control problem.

**Problem 2:** Suppose that for the system (1), the initial time $t_0$, the initial state $x(0) = x_0$, $h_0$ and $h_1 = h_2 = \cdots = h$, $N \geq 1$ are given. Then find a control input sequence $u_0, u_1, \ldots, u_{N-1}$ minimizing the cost function (2) under the input constraint (3) ■

From the result on sampled-data control theory, Problem 2 can be equivalently rewritten as the following optimal control problem of discrete-time linear systems.

**Problem 3:** Suppose that the initial time $t_0$, the initial state $x(0) = x_0$, $h_0$ and $h_1 = h_2 = \cdots = h$, $N \geq 1$ are given. Consider the following discrete-time linear system

$$x_1 = \tilde{A}(h_0)x_0 + \tilde{B}(h_0)u_0,$$

$$x_{k+1} = \tilde{A}(h)x_k + \tilde{B}(h)u_k, \ \ k \geq 1$$

where $x_k := x(t_k)$. Then find a control input sequence $u_0, u_1, \ldots, u_{N-1}$ minimizing the cost function (2), i.e.,

$$J = \begin{bmatrix} x_0 \\ u_0 \end{bmatrix}^T \begin{bmatrix} \tilde{Q}(h_0) & \tilde{S}(h_0) \\ \tilde{S}^T(h_0) & \tilde{R}(h_0) \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} + \sum_{k=1}^{N-1} \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T \begin{bmatrix} \tilde{Q}(h) & \tilde{S}(h) \\ \tilde{S}^T(h) & \tilde{R}(h) \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + x_N^T P(h) x_N \tag{5}$$

under the input constraint (3). ■

Next, consider reducing Problem 3 to a QP problem. Define

$$\bar{x} := \left[ \, x_0^T \ x_1^T \ \cdots \ x_N^T \, \right]^T,$$

$$\bar{u} := \left[ \, u_0^T \ u_1^T \ \cdots \ x_{N-1}^T \, \right]^T.$$

Then we can obtain $\bar{x} = \bar{A}x_0 + \bar{B}\bar{u}$ where

$$\bar{A} = \begin{bmatrix} I \\ \tilde{A}(h_0) \\ \tilde{A}(h)\tilde{A}(h_0) \\ \tilde{A}^2(h)\tilde{A}(h_0) \\ \vdots \\ \tilde{A}^{N-1}(h)\tilde{A}(h_0) \end{bmatrix}$$

and

$$\bar{B} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \tilde{B}(h_0) & 0 & \cdots & \vdots \\ \tilde{A}(h)\tilde{B}(h_0) & \tilde{B}(h) & \cdots & \vdots \\ \tilde{A}^2(h)\tilde{B}(h_0) & \tilde{A}(h)\tilde{B}(h) & \cdots & \vdots \\ \vdots & \vdots & \cdots & 0 \\ \tilde{A}^{N-1}(h)\tilde{B}(h_0) & \tilde{A}^{N-2}(h)\tilde{B}(h) & \cdots & \tilde{B}(h) \end{bmatrix}.$$

In addition, we define

$$\bar{Q} := \text{block-diag}(\tilde{Q}(h_0), \tilde{Q}(h), \ldots, \tilde{Q}(h), P(h)),$$

$$\bar{S} := \begin{bmatrix} \text{block-diag}(\tilde{S}(h_0), \tilde{S}(h), \ldots, \tilde{S}(h)) \\ 0_{n\times(N-1)m} \end{bmatrix},$$

$$\bar{R} := \text{block-diag}(\tilde{R}(h_0), \tilde{R}(h), \ldots, \tilde{R}(h)).$$

Then the cost function (5) can be rewritten as follows:

$$\begin{aligned} J &= \bar{x}^T \bar{Q} \bar{x} + 2\bar{x}^T \bar{S} \bar{u} + \bar{u}^T \bar{R} \bar{u} \\ &= \bar{u}^T L_2 \bar{u} + L_1 \bar{u} + L_0 \end{aligned}$$

where

$$\begin{aligned} L_2 &= \bar{R} + \bar{B}^T \bar{S} + \bar{S}^T \bar{B} + \bar{B}^T \bar{Q} \bar{B}, \\ L_1 &= 2x_0^T \bar{A}^T (\bar{S} + \bar{Q}\bar{B}), \\ L_0 &= x_0^T \bar{A}^T \bar{Q} \bar{A} x_0. \end{aligned}$$

Finally, $\bar{u}_{\min} := [\ u_{\min}^T\ u_{\min}^T\ \cdots\ u_{\min}^T\ ]^T$ and $\bar{u}_{\max} := [\ u_{\max}^T\ u_{\max}^T\ \cdots\ u_{\max}^T\ ]^T$ are also defined.

Under the above preparation, Problem 3 is equivalent to the following QP problem:

**Problem A:**

$$\begin{aligned} \text{find} \quad & u_0, u_1, \ldots, u_{N-1}, \\ \min \quad & \bar{u}^T L_2 \bar{u} + L_1 \bar{u} + L_0, \\ \text{subject to} \quad & \bar{u}_{\min} \le \bar{u} \le \bar{u}_{\max}. \end{aligned}$$

A QP problem can be solved by using a suitable solver such as MATLAB and IBM ILOG CPLEX [15].

Finally, by using the obtained QP problem, we propose an algorithm for solving Problem 1.

**Algorithm 1:**

**Step 1:** Derive $J_h^*$ by solving Problem A with $h_0 = h$.

**Step 2:** Set $a = h$ and $b = h_{\max}$, and give a sufficiently small positive real number $\varepsilon$.

**Step 3:** Set $h_0 = (a + b)/2$.

**Step 4:** Derive $J_{h_0}^*$ by solving Problem A.

**Step 5:** If $J_{h_0}^* \le \gamma J_h^*$ in Problem 1 is satisfied, then set $a = h_0$, otherwise set $b = h_0$.

**Step 6:** If $|a - b| < \varepsilon$ is satisfied, then the optimal $h_0$ in Problem 1 is derived as $a$, and the optimal control input sequence is also derived. Otherwise go to Step 3.

In a numerical example (Section 5), we will discuss the computation time of Algorithm 1.

# 4. SELF-TRIGGERED MODEL PREDICTIVE CONTROL

We show a procedure of MPC based on the proposed solution method for Problem 1.

**Procedure of Self-Triggered MPC:**

**Step 1:** Set $t_0 = 0$, and give the initial state $x(0) = x_0$.

**Step 2:** Solve Problem 1.

**Step 3:** Apply only $u(t), t \in [t_0, t_0 + h_0)$ to the plant.

**Step 4:** Compute the predicated state $\hat{x}(t_0 + h_0)$ by using $x(t_0)$, $h_0$ and $u_0$.

**Step 5:** Solve Problem 1 by using $\hat{x}(t_0 + h_0)$ as $x_0$.

**Step 6:** Wait until time $t_0 + h_0$.

**Step 7:** Update $t_0 := t_0 + h_0$, and measure $x(t_0)$. Return to Step 3.

Note here that in this procedure, the timing (i.e., the sampling time) to measure the state and to recompute the control input is computed. In this sense, self-triggered control is realized.

# 5. NUMERICAL EXAMPLE

Consider the following system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -5 & -8 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{6}$$

The input constraint is given as $u(t) \in [-10, +10]$. Parameters in Problem 1 are given as follows: $h = 0.5$, $\gamma = 1.001$, $h_{\max} = 5$, $Q = 10^3 I_n$, and $R = 1$. In Algorithm 1, we set $\varepsilon = 10^{-4}$. Then $P(h)$ can be derived as

$$P(h) = \begin{bmatrix} 10615 & 593 \\ 593 & 575 \end{bmatrix}.$$

In addition, we consider two cases, i.e., the case of $N = 1$ and the case of $N = 10$.

We show the computation result on self-triggered MPC using Problem 1. The initial state is given as $x_0 = [\ 10\ 10\ ]^T$, and the case of $N = 10$ is considered. Fig. 1 shows the obtained state and input trajectories. From this figure, we see that the sampling period is non-uniform.

Next, compare two cases. In these cases, the obtained state trajectories are almost the same. The difference between two cases is as follows. In Fig. 1, the value of the control input at each time is shown as follows:

$$\begin{aligned} u(t) &= -10.00, \ \ t \in [0, 1.83), \ \ h_0 = 1.83, \\ u(t) &= -1.90, \ \ t \in [1.83, 3.23), \ \ h_0 = 1.41, \\ u(t) &= -0.49, \ \ t \in [3.23, 4.49), \ \ h_0 = 1.25, \\ u(t) &= -0.14, \ \ t \in [4.49, 5.69), \ \ h_0 = 1.21, \\ u(t) &= -0.04, \ \ t \in [5.69, 6.89), \ \ h_0 = 1.20, \\ u(t) &= -0.01, \ \ t \in [6.89, 8.08), \ \ h_0 = 1.19. \end{aligned}$$

On the other hand, in the case of $N = 1$, the value of the control input at each time is derived as follows:

$$\begin{aligned} u(t) &= -10.00, \ \ t \in [0, 0.62), \ \ h_0 = 0.62, \\ u(t) &= -10.00, \ \ t \in [0.62, 1.66), \ \ h_0 = 1.03, \\ u(t) &= -2.62, \ \ t \in [1.66, 2.88), \ \ h_0 = 1.22, \\ u(t) &= -0.75, \ \ t \in [2.88, 4.08), \ \ h_0 = 1.20, \\ u(t) &= -0.22, \ \ t \in [4.08, 5.27), \ \ h_0 = 1.19, \\ u(t) &= -0.06, \ \ t \in [5.27, 6.47), \ \ h_0 = 1.19, \\ u(t) &= -0.02, \ \ t \in [6.47, 7.66), \ \ h_0 = 1.19, \\ u(t) &= -0.01, \ \ t \in [7.66, 8.85), \ \ h_0 = 1.19. \end{aligned}$$

From these results, we can obtain the following observation. In this example, input saturation is needed to improve the transient behavior. However, in the case of $N = 1$, the time interval of input saturation was not computed suitably. As a result, to derive the state trajectory

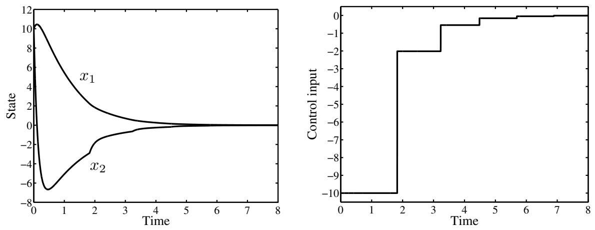


Fig. 1 State and input trajectories ($N = 10$).

at time interval $[0, 8)$, Problem 1 was solved eight times. In the case of $N = 10$, Problem 1 was solved six times, and the above problem is overcome. So it is important to choose a suitable $N$. We remark that in this example, the computation result in the case of $N = 20$ is the same as that in the case of $N = 10$. In this sense, $N = 10$ is one of the suitable horizons.

In addition, we discuss the effect of changing $\gamma$ in (4). In the case of $N = 10$, consider the following cases: $\gamma = 1.001$, $1.005$, $1.010$, $1.015$, $1.020$. For each case, the first $h_0$ is obtained as follows:

$$\gamma = 1.001 : \quad h_0 = 1.83,$$
$$\gamma = 1.005 : \quad h_0 = 2.38,$$
$$\gamma = 1.010 : \quad h_0 = 2.79,$$
$$\gamma = 1.015 : \quad h_0 = 3.12,$$
$$\gamma = 1.020 : \quad h_0 = 3.44.$$

From these results, we see that $h_0$ becomes longer by setting a larger $\gamma$. Since control performance decreases for a larger $\gamma$, it is important to consider the trade-off between $\gamma$ and $h_0$.

Finally, we discuss the computation time for solving Problem 1. In the case of $N = 10$, Problem 1 with the different initial state was solved six times. Then the mean computation time for solving Problem 1 was 6.51 [sec], where we used IBM ILOG CPLEX 11.0 [15] as the MIQP solver on the computer with the Intel Core2 Duo 3.0GHz processor and the 2GB memory. In the case of $N = 1$, Problem 1 with the different initial state was solved eight times. Then the mean computation time was 6.22 [sec]. It is one of the future works to consider several approaches for reducing the computation time.

## 6. CONCLUSION

In this paper, we have proposed a self-triggered optimal control method based on model predictive control. By focusing on only the first sampling period in the optimal control problem, we have proposed an iterative solution method (Algorithm 1). The effectiveness of the proposed method has been shown by a numerical example. The proposed methods are useful as a new method of self-triggered optimal control.

In the future works, it is important to develop a more efficient method for solving Problem 1. Then continuation methods [11] may be useful. It is also significant to analyze stability of the closed-loop system.

This work was partially supported by Grant-in-Aid for Young Scientists (B) 23760387.

## REFERENCES

[1] Special section on complex networked control systems, *IEEE Control Systems Magazine*, Vol. 27, No. 4, 2007.

[2] A. Anta and P. Tabuada, Self-triggered stabilization of homogeneous control systems, *Proc. American Control Conf.*, pp. 4129–4134, 2008.

[3] A. Anta and P. Tabuada, To sample or not to sample: Self-triggered control for nonlinear systems, *IEEE Trans. on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, 2009.

[4] P. J. Antsaklis and J. Baillieul (Guest Editors), Special issue on the technology of networked control systems, *Proceedings of the IEEE*, Vol. 95, No. 1, 2007.

[5] A. Camacho, P. Martí, M. Velasco, C. Lozoya, R. Villá, J. M. Fuertes, and E. Griful, Self-triggered networked control systems: an experimental case study, *Proc. IEEE Int'l Conf. on Industrial Technology*, pp. 123–128, 2010.

[6] L.-S. Hu, T. Bai, P. Shi, and Z. Wu, Sampled-data control of networked linear control systems, *Automatica*, Vol. 43, No. 5, pp. 903–911, 2007.

[7] H. Ishii, Stabilization under shared communication with message losses and its limitations, *Proc. 45th IEEE Conf. on Decision and Control*, pp. 4974–4979, 2006.

[8] H. Ishii, $H^\infty$ control with limited communication and message losses, *Systems & Control Letters*, Vol. 57, No. 4, pp. 322–331, 2008.

[9] K. Kobayashi and K. Hiraishi, Optimal control of a class of networked systems based on MLD framework, *Proc. the 18th IFAC World Congress*, pp. 66–71, 2011.

[10] M. Mazo Jr. and P. Tabuada, On event-triggered and self-triggered control over sensor-actuator networks, *Proc. 47th IEEE Conf. on Decision and Control*, pp. 435–440, 2008.

[11] T. Otsuka, A continuation/GMRESmethod for fast computation of nonlinear receding horizon control, *Automatica*, vol. 40, pp. 563–574, 2004.

[12] M. Velasco, P. Martí, J. Yépez, F. J. Ruiz, J. M. Fuertes, and E. Bini, Qualitative analysis of a one-step finite horizon boundary for event-driven controllers, *Proc. 50th IEEE Conf. on Decision and Control and European Control Conf.*, pp. 1662–1667, 2011.

[13] X. Wang and M. D. Lemmon, Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability, *IEEE Trans. on Automatic Control*, vol. 54, no. 3, pp. 452–467, 2009.

[14] J. Yépez, M. Velasco, P. Martí, E. X. Martín, and J. M. Fuertes, One-step finite horizon boundary with varying control gain for event-driven networked control systems, *Proc. 37th Annual Conf. of the IEEE Industrial Electronics Society*, pp. 2531–2536, 2011.

[15] http://www.ilog.com/products/cplex/