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Discrete-Time Iterative Learning Control for Relative Degree Systems: A 2-D Approach

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Abstract: This paper is devoted to the two-dimensional (2-D) design problem that arises from discrete-time iterative learning control (ILC). For linear time-invariant (LTI) systems with well-defined relative degree, a unified ILC algorithm is considered which provides wider freedom for the updating law formation. It demonstrates that an appropriately defined variable, together with the tracking error, can be employed to establish the Roesser systems based 2-D description of the ILC process. This enables both asymptotic stability and monotonic convergence of the relative degree ILC systems to be achieved. In particular, conditions for the monotonic convergence are described in terms of linear matrix inequalities (LMIs), which directly give formulas for the updating law design.

Keywords: Discrete-time iterative learning control; relative degree; monotonic convergence; linear matrix inequality.

I. INTRODUCTION

Iterative learning control (ILC) is known as an effect technique for systems operating repetitively over a fixed time interval. The key feature of ILC is a fundamentally two-dimensional (2-D) process [1], with evolution in two independent directions. In order to take into account the entire dynamics of an ILC, the 2-D analysis approach is found to be a good alternative which can be implemented based on the Roesser's type 2-D systems (see, e.g., [2]-[7]). Hence, the well-developed theory of 2-D systems can be employed to deal with the ILC design. However, the existing 2-D analysis approach is only applicable to the ILC design for such systems with relative degree that is not more than one. This is because it is difficult to establish the 2-D model description of ILC systems with higher-order relative degree. In fact, the system relative degree plays a significant role not only in the ILC design but also in the ILC convergence analysis [8]. Moreover, the area of monotonically convergent ILC design has seen relatively little activity when it comes to addressing systems with higher-order relative degree.

In this paper, the 2-D design approach is investigated for discrete-time ILC with relative degree. It shows that the 2-D Roesser systems can be established to describe the entire dynamics involved in ILC with well-defined system relative degree. Based on the 2-D system theory, a convergence analysis of ILC can be directly presented, and a necessary and sufficient condition can be provided, which is dependent only upon the first non-zero Markov parameter matrix. Moreover, after giving the relationship between two sequential iteration tracking errors from the 2-D Roesser systems, the monotonic convergence of ILC can be obtained by applying the bounded real lemma [9]. In this case, it shows that the monotonically convergent ILC can be designed through the linear matrix inequality (LMI) technique, and formulas can be presented for the control law design. Finally, a simulation test is proposed to illustrate that the 2-D approach can be used to address monotonically convergent ILC with relative degree.

Notations: I and 0 denote the identity matrix and the zero matrix with required dimensions, respectively; $M > 0$ (respectively, $M < 0$) denotes a symmetric positive (respectively, negative) definite matrix; an asterisk ($\bigstar$) denotes a term that is induced by symmetry. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations. For a given vector $x_k(t)$, let q be a shift operator such that $q : x_k(t) \rightarrow qx_k(t) = x_k(t+1)$, and Δ be a difference operator such that $\Delta : x_k(t) \rightarrow \Delta x_k(t) = x_{k+1}(t) - x_k(t)$.

II. ILC SYSTEM DESCRIPTION

Consider the system over $t \in \{0,\ 1,\ \cdots,\ T-1\}$ (short for $t \in [0, T-1]$) and $k \in \mathbb{Z}_+$:

$$\begin{aligned} x_k(t+1) &= Ax_k(t) + Bu_k(t) \\ y_k(t) &= Cx_k(t), \quad x_k(0) = x_0,\ \forall k \end{aligned} \tag{1}$$

where $x_k(t) \in \mathbb{R}^n$ is the state, $u_k(t) \in \mathbb{R}^m$ is the input, $y_k(t) \in \mathbb{R}^l$ is the output, and $(A,\ B,\ C)$ is the constant system matrix pair of appropriate dimensions.

It is assumed that system (1) has a relative degree of $r \geq 1$ which is defined as follows.

- *Relative Degree:* The relative degree r of system (1) is an integer which can be characterized by the following conditions:
  1) $CA^iB = 0$ for all $i < r-1$;
  2) $CA^{r-1}B \neq 0$ and is of full-row rank.

To deal with the relative degree r, the ILC considered in this paper uses an updating law given by

$$u_{k+1}(t) = u_k(t) + \sum_{i=0}^{r} K_i q^i e_k(t) \tag{2}$$

where $e_k(t) = y_d(t) - y_k(t)$ is the tracking error, and K_i, $i = 0, 1, \ldots, r$, is an $m \times l$ gain matrix to be designed. The trajectory $y_d(t)$ is the desired output to be tracked over $[r, T + r]$.

The objective of this paper is to address convergence and design problems of the ILC system (1) and (2) by developing a 2-D approach under the Roesser systems framework. To this end, we assume that the initial reset condition is satisfied, i.e., $q^i y_k(0) = q^i y_d(0)$ and, without loss of generality, $q^i y_k(0) = q^i y_d(0) = 0$ is considered, where $i = 0, \ldots, r-1$.

III. MAIN RESULTS

A. 2-D System Representation

First of all, the 2-D Roesser systems based approach will be developed for ILC with a system relative degree $r \geq 1$. To this end, let us denote

$$\begin{cases} \eta_{1k}(t) = A^r q^{-r} \Delta x_k(t) \\ \eta_{2k}(t) = \sum_{j=0}^{r-1} \sum_{i=0}^{j} A^j B K_i q^{-(j-i)-1} e_k(t) \\ \eta_{3k}(t) = \sum_{j=0}^{r-1} \sum_{i=j+1}^{r} A^j B K_i q^{i-j-1} e_k(t) \\ \eta_{4k}(t) = \sum_{j=0}^{r-1} A^r B K_j q^{-r+j} e_k(t) \\ \eta_{5k}(t) = \sum_{j=0}^{r-2} \sum_{i=j}^{r-2} A^{i+1} B K_{i-j} q^{-j-1} e_k(t) \end{cases} \tag{3}$$

where $\sum_{j=i}^{i-1}(\cdot)_j \triangleq 0$, $\forall i$. Particularly, $\eta_{5k}(t) = 0$ always holds when $r = 1$. For the variables of (3), some properties are given as follows.

Lemma 1: Consider $\eta_{ik}(t)$ defined in (3), where $i = 2$, 3, 4, and 5. Then

a) $\eta_{2k}(t)$ can be rewritten as

$$\eta_{2k}(t) = \sum_{j=0}^{r-1} \sum_{i=j}^{r-1} A^i B K_{i-j} q^{-j-1} e_k(t). \tag{4}$$

b) $\eta_{3k}(t)$ can be such that

$$C\eta_{3k}(t) = CA^{r-1} B K_r e_k(t). \tag{5}$$

c) $\eta_{4k}(t)$ and $\eta_{5k}(t)$ satisfy

$$\eta_{4k}(t) + \eta_{5k}(t) = A\eta_{2k}(t). \tag{6}$$

Proof: The proof can be immediately derived with some algebraic operation and, thus, is omitted here. ■

Now with Lemma 1, let us consider the 2-D representation of ILC systems. The use of $e_k(t) = y_d(t) - y_k(t)$ and $y_k(t) = Cx_k(t)$ leads to

$$\begin{aligned} e_{k+1}(t) &= \Delta e_k(t) + e_k(t) \\ &= \Delta [y_d(t) - y_k(t)] + e_k(t) \\ &= -\Delta y_k(t) + e_k(t) \\ &= -C\Delta x_k(t) + e_k(t). \end{aligned} \tag{7}$$

where $\Delta x_k(t)$, by using the first equation of system (1), satisfies

$$\begin{aligned} q\Delta x_k(t) &= \Delta q x_k(t) \\ &= \Delta x_k(t+1) \\ &= \Delta [Ax_k(t) + Bu_k(t)] \\ &= A\Delta x_k(t) + B\Delta u_k(t). \end{aligned} \tag{8}$$

Since $\Delta x_k(0) = 0$, the use of (8) can yield

$$\Delta x_k(t) = Aq^{-1}\Delta x_k(t) + Bq^{-1}\Delta u_k(t). \tag{9}$$

Following the same steps repetitively, it can be developed further from (9) that

$$\begin{aligned} \Delta x_k(t) &= A^r q^{-r} \Delta x_k(t) + \sum_{j=0}^{r-1} A^j B q^{-j-1} \Delta u_k(t) \\ &= \eta_{1k}(t) + \sum_{j=0}^{r-1} A^j B q^{-j-1} \Delta u_k(t). \end{aligned} \tag{10}$$

The ILC law of (2) can be rewritten as

$$\Delta u_k(t) = \sum_{i=0}^{r} K_i q^i e_k(t). \tag{11}$$

Then insert (11) into (10) to get

$$\begin{aligned} \Delta x_k(t) &= \eta_{1k}(t) + \sum_{j=0}^{r-1} A^j B q^{-j-1} \sum_{i=0}^{r} K_i q^i e_k(t) \\ &= \eta_{1k}(t) + \sum_{j=0}^{r-1} \sum_{i=0}^{r} A^j B K_i q^{i-j-1} e_k(t) \\ &= \eta_{1k}(t) + \sum_{j=0}^{r-1} \sum_{i=0}^{j} A^j B K_i q^{-(j-i)-1} e_k(t) \\ &\quad + \sum_{j=0}^{r-1} \sum_{i=j+1}^{r} A^j B K_i q^{i-j-1} e_k(t) \\ &= \eta_{1k}(t) + \eta_{2k}(t) + \eta_{3k}(t). \end{aligned} \tag{12}$$

In view of (5) and by inserting (12) into (7), it follows immediately that

$$\begin{aligned} e_{k+1}(t) &= -C[\eta_{1k}(t) + \eta_{2k}(t)] - C\eta_{3k}(t) + e_k(t) \\ &= -C\xi_k(t) + \left(I - CA^{r-1} B K_r\right) e_k(t) \end{aligned} \tag{13}$$

where

$$\xi_k(t) = \eta_{1k}(t) + \eta_{2k}(t). \tag{14}$$

From (13), it is obvious that an iterative equation about the tracking error is obtained, which can reflect the ILC system dynamics along the iteration axis k. With this fact, $\xi_k(t)$ will be further discussed in order to disclose the time-domain dynamics involved in the ILC system (1) and (2).

To describe the ILC system dynamics along the time axis t, next let us consider $\xi_k(t+1) = q\xi_k(t)$. Towards this end, compute $q\eta_{1k}(t)$ and then insert (8) and (11) to obtain

$$\begin{aligned} q\eta_{1k}(t) &= q\left[A^r q^{-r}\Delta x_k(t)\right] \\ &= A^r q^{-r} q\Delta x_k(t) \\ &= A^{r+1} q^{-r}\Delta x_k(t) + A^r B q^{-r}\Delta u_k(t) \\ &= A\eta_{1k}(t) + A^r B q^{-r}\sum_{j=0}^{r} K_j q^j e_k(t) \\ &= A\eta_{1k}(t) + \sum_{j=0}^{r-1} A^r B K_j q^{-r+j} e_k(t) + A^r B K_r e_k(t) \\ &= A\eta_{1k}(t) + \eta_{4k}(t) + A^r B K_r e_k(t). \end{aligned} \tag{15}$$

And with (4), $q\eta_{2k}(t)$ is computed by

$$\begin{aligned} q\eta_{2k}(t) &= q\sum_{j=0}^{r-1}\sum_{i=j}^{r-1} A^i B K_{i-j} q^{-j-1} e_k(t) \\ &= \sum_{j=1}^{r-1}\sum_{i=j}^{r-1} A^i B K_{i-j} q^{-j} e_k(t) + \sum_{i=0}^{r-1} A^i B K_i e_k(t) \\ &= \sum_{j=0}^{r-2}\sum_{i=j}^{r-2} A^{i+1} B K_{i-j} q^{-j-1} e_k(t) + \sum_{i=0}^{r-1} A^i B K_i e_k(t) \\ &= \eta_{5k}(t) + \sum_{i=0}^{r-1} A^i B K_i e_k(t). \end{aligned} \tag{16}$$

Use (6), (15) and (16) to obtain

$$\begin{aligned} q\xi_k(t) &= q\eta_{1k}(t) + q\eta_{2k}(t) \\ &= A\eta_{1k}(t) + \eta_{4k}(t) + \eta_{5k}(t) + \sum_{i=0}^{r} A^i B K_i e_k(t) \\ &= A\left[\eta_{1k}(t) + \eta_{2k}(t)\right] + \sum_{i=0}^{r} A^i B K_i e_k(t) \\ &= A\xi_k(t) + \sum_{i=0}^{r} A^i B K_i e_k(t). \end{aligned} \tag{17}$$

Thus, based on (13) and (17), the following 2-D Roesser model can be established:

$$\begin{bmatrix} \xi_k(t+1) \\ e_{k+1}(t) \end{bmatrix} = \begin{bmatrix} A & \sum_{i=0}^{r} A^i B K_i \\ -C & I - CA^{r-1}BK_r \end{bmatrix} \begin{bmatrix} \xi_k(t) \\ e_k(t) \end{bmatrix} \tag{18}$$

which clearly describes the two independent dynamics involved in the ILC system (1) and (2), as claimed in [2]-[4], [6]. For the 2-D model of (18), the use of (3) and (14) yields

$$\begin{aligned} \xi_k(t) &= A^r q^{-r}\Delta x_k(t) + \sum_{j=0}^{r-1}\sum_{i=0}^{j} A^j B K_i q^{-(j-i)-1} e_k(t) \\ &= A^r \Delta x_k(t-r) + \sum_{j=0}^{r-1}\sum_{i=0}^{j} A^j B K_i e_k(t-(j-i)-1). \end{aligned}$$

Note that $\Delta x_k(0) = 0$ holds, and $q^i y_k(0) = q^i y_d(0)$ implies $y_k(i) = y_d(i)$, i.e., $e_k(i) = 0$, for $i = 0, 1, \ldots, r-1$. Hence, it is obvious that $\xi_k(r) = 0$ holds for $k \in \mathbb{Z}_+$. Consequently, the boundary conditions for (18) are:

$$\xi_k(r) = 0 \text{ for } k \in \mathbb{Z}_+ \text{ and finite } e_0(t) \text{ for } t \in [r, T+r]. \tag{19}$$

Remark 1: From (18), it is clear that Roesser models can be developed with $\xi_k(t)$ and $e_k(t)$ to describe the 2-D processes resulting from the ILC system (1) and (2). This implies that a general 2-D framework is established for ILC systems with relative degree. For the particular case where $r = 1$, it can be easily shown that the 2-D Roesser system (18) provides an alternative approach to describe the ILC systems that have been considered in, e.g., [2]-[4].

B. Convergence Analysis of ILC

With the development of 2-D system representation, both asymptotic stability and monotonic convergence can be considered for ILC systems with relative degree. First, the following result is given for the asymptotic stability of ILC.

Proposition 1: Consider the ILC system (1) and (2) of the general relative degree $r \geq 1$. Then the tracking error $e_k(t)$ converges asymptotically to zero as $k \to \infty$ if and only if the matrix $I - CA^{r-1}BK_r$ is stable, i.e., the spectral radius fulfills $\rho\left(I - CA^{r-1}BK_r\right) < 1$.

Proof: With the 2-D system theory applied to (18) and (19), the proof is immediate. For more details, see [3] and [4]. ■

Remark 2: From Proposition 1, it is obvious that the asymptotic stability of ILC depends only upon the first non-zero Markov parameter matrix $CA^{r-1}B$, regardless of the system relative degree r.

Next, the following result is presented for the monotonic convergence of ILC.

Proposition 2: Consider the ILC system (1) and (2) of the general relative degree $r \geq 1$. Then the tracking error $e_k(t)$ converges monotonically to zero when $k \to \infty$ in the sense of the $\mathscr{L}_2$-norm if there exist scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and matrices $Q > 0$, X_i, $i = 0, \ldots, r$, that satisfy the following LMIs

$$\varepsilon_1 \leq \varepsilon_2 \tag{20}$$

$$\begin{bmatrix} -Q & (\star) & (\star) & (\star) \\ QA^{\mathrm{T}} & -Q & (\star) & (\star) \\ \sum_{i=0}^{r} X_i^{\mathrm{T}} B^{\mathrm{T}} A^{i^{\mathrm{T}}} & 0 & -\varepsilon_1 I & (\star) \\ 0 & CQ & \varepsilon_2 I + CA^{r-1}BX_r & -\varepsilon_1 I \end{bmatrix} < 0. \tag{21}$$

If the LMIs of (20) and (21) are feasible, then the gain matrices are given by

$$K_i = -\varepsilon_2^{-1} X_i, \quad i = 0, \ldots, r. \tag{22}$$

Proof: With the bounded real lemma (see, e.g., [9]) applied, the proof can be proved based on the use of (18) and in the same way as in the proof of [8, Theorem 3] and, thus, is omitted here. ■

Remark 3: Proposition 2 implies that although the asymptotic stability of the updating law (2) depends only on the selection of K_r, its other learning gains can help to achieve the monotonic convergence of ILC to ensure good performance.

IV. NUMERICAL EXAMPLE

In this example, system (1) is considered with matrices given by:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -0.0043 & 0.0004 & 0.12 & 0.149 & -0.71 & -1.7 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Clearly, it is easy to show that $CB = 0$ and CAB has full-row rank, resulting in system (1) with a relative degree of $r = 2$.

To perform the simulation, the zero initial control input $u_0(t) = 0$ is used, and the following desired trajectory is considered:

$$y_d(t) = \begin{bmatrix} y_{d1}(t) \\ y_{d2}(t) \end{bmatrix} = \begin{bmatrix} 20 - 20\cos(0.02\pi t) \\ 6\cdot 10^{-10}t^5 - 1.5\cdot 10^{-7}t^4 + 10^{-5}t^3 \end{bmatrix}$$

where $t \in [0, 100]$. Accordingly, $y_k(t) = [y1_k(t),\ y2_k(t)]^{\mathrm{T}}$ is denoted for the sake of notations. Then solve LMIs (20) and (21) with $r = 2$ to derive

$$K_0 = \begin{bmatrix} 0.0434 & 0.1665 \\ -0.0119 & 2.4293 \\ 0.0078 & -1.0000 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0.2855 & -0.2664 \\ -0.0354 & 1.1861 \\ -0.0007 & -2.4293 \end{bmatrix}$$

$$K_2 = \begin{bmatrix} 0.5883 & -0.0000 \\ -0.3300 & 0.2664 \\ 0.0309 & -1.3525 \end{bmatrix}.$$

Now using such gain matrices, we perform the ILC system (1) and (2), and show the test results in Figs. 1 and 2. Fig. 1 shows the evolution of tracking errors $\|y_{d1}(t) - y1_k(t)\|_2$ and $\|y_{d2}(t) - y2_k(t)\|_2$ with respect to the iteration number k, and Fig. 2 shows the time evolution of the reference trajectory $y_d(t)$ and actual output $y_k(t)$ for $k = 1, 3, 5$. From Figs. 1 and 2, it is clear that the ILC process converges monotonically. It illustrates that the proposed 2-D approach can effectively address the design of ILC with system relative degree.

V. CONCLUSIONS

In this paper, the 2-D design approach to discrete-time ILC with relative degree has been discussed. It has been shown that the asymptotic stability of ILC is dependent only upon the first non-zero Markov parameter matrix. Moreover, sufficient conditions have been provided in terms of LMIs to guarantee the monotonic convergence of ILC and give formulas for the updating law design. For ILC designed through solving LMIs, its effectiveness has been verified finally through simulation test.

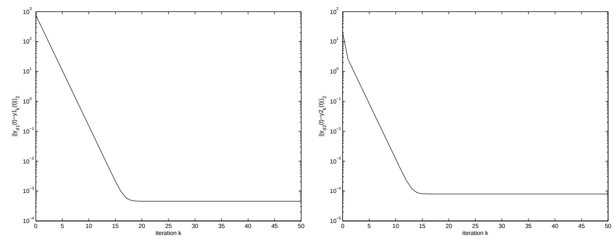

Fig.1. Left: The error between $y_{d1}(t)$ and $y1_k(t)$.
Right: The error between $y_{d2}(t)$ and $y2_k(t)$.

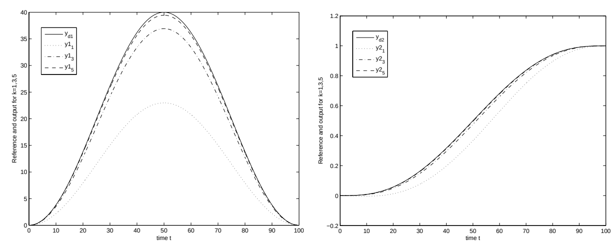

Fig.2. Left: $y_{d1}(t)$ and $y1_k(t)$ for $k = 1, 3, 5$.
Right: $y_{d2}(t)$ and $y2_k(t)$ for $k = 1, 3, 5$.

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