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Feedback Stabilization of Linear Systems with Distributed Input Time-delay by Backstepping Method

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Abstract: The stabilization problem of a linear time-invariant system with lumped and distributed delays in the control will be investigated by the backstepping method. A transformation is introduced firstly to reduce the system with distributed input delay into a system with lumped input delay. The transformation kernel can be expressed explicitly through solving a Cauchy problem of ODEs. Then the backstepping arguments presented by Krstic and Smyshlyaev in [1] can be applied to work out a feedback control for the original system, where the key point is to model the lumped delay by a first-order hyperbolic partial differential equation.

Keywords: time-delay systems, distributed delay, backstepping method, reduction transformation.

I. INTRODUCTION

In many engineering systems, the variation of the system state depends on past states. Such a character is called time-delay. Time-delay systems are also known as systems with aftereffect or dead-time, hereditary systems, which have attracted the attention of many researchers because of their importance and widespread occurrence. They are special infinite dimensional systems, and researches indicate that the time delays lead to some complexity. On one hand, the time delays may deteriorate the control performance or even cause the instability of a dynamic system. On the other hand, several studies have shown that voluntary introduction of delays can also benefit the control.

The stabilization for time-delay systems is a topic of great importance and has received increasing attention [2], [3], [4], [5]. Smith-predictor is well-known for designing linear feedback controllers for systems with delay. However, this method is confined to stable plants because it implies pole-zero cancellations. Moreover, Smith predictor is sensitive to parameter errors. The model reduction method is another important approach to deal with the systems with input delays, where the so-called Artstein model reduction is often involved. Briefly speaking, the model reduction is a transformation through which one can simplify a dynamic system with input delay into an equivalent delay-free system. Considering the following linear system with lumped input delay,

$$\dot{x}(t) = Ax(t) + B_0 u(t) + B_1 u(t-\tau), \tag{1}$$

introduce the new variable $y(t)$ defined by

$$y(t) = x(t) + e^{-A\tau} \int_{t-\tau}^{t} e^{A(t-s)} B_1 u(s) ds. \tag{2}$$

Then (1) is reduced to a delay-free system

$$\dot{y}(t) = Ay(t) + \left(B_0 + e^{-A\tau} B_1\right) u(t).$$

It is straightforward to compute a state feedback control $u(t) = Ky(t)$, provided that $(A, B_0 + e^{-A\tau} B_1)$ is stabilizable. Thus the system (1) can be stabilized by the control law

$$u(t) = K \quad x(t) + e^{-A\tau} \int_{t-\tau}^{t} e^{A(t-s)} B_1 u(s) ds \quad . \tag{3}$$

Kwon and Pearson were the first who clearly put forward the reduction transformation (2) for a system with lumped input delay [4]. In [4], the authors applied the receding horizon method to the stabilization of linear time-invariant systems with lumped input delay. It was shown how the receding horizon control suggested the reduction transformation (2). Afterwards, Artstein established the general abstract theory of the reduction method in [2]. According to this theory, one can transform general linear time-varying systems into delay-free systems. It has been shown by many researches that the reduction provides a strong tool for manipulating systems with delays in the controls, even for linear systems with time-varying lumped input delay [2].

It has also been pointed out in [5] that any lumped time-delay $U(t) = u(t-\tau)$ can be represented by a classical transport equation:

$$\begin{cases} \tau \frac{\partial}{\partial t} v(x,t) + \frac{\partial}{\partial x} v(x,t) = 0, & x \in [0,1], \\ v(0,t) = u(t), \quad v(1,t) = U(t). \end{cases}$$

In other words, we can use a first-order hyperbolic partial differential equation to model the lumped delay. A problem of boundary feedback stabilization of first-order hyperbolic partial differential equations was considered in [1], where the authors applied the backstepping method to design controllers. As an example of the applications, the authors studied the stabilization of a linear time-invariant system with lumped input delay by combining the backstepping design for hyperbolic PDEs with the backstepping design for linear ODEs.

Motivated by the researches mentioned above, we will investigate the stabilization problem of a linear time-invariant system with lumped and distributed delays in the control by the backstepping method. We first transform the linear system with distributed input delay into a system with lumped input delay, where the transformation kernel is determined by a Cauchy problem of ODEs. This transformation kernel can be expressed explicitly. Then the procedure presented in [1] can be applied in our circumstance. Concretely, we model the lumped delay by a first-order hyperbolic partial differential equation, and replace the resulting system in the first step by an ODE-PDE cascade. After introducing a backstepping transformation, we obtain the target system whose origin is exponentially stable. Thus we can work out the corresponding feedback control law for the original system.

The rest of this paper is organized as follows. In section 2, the linear time-invariant system with lumped and distributed delay under consideration is given. The feedback stabilization control law is deduced in detail based on an integral transformation and the backstepping method, and the main result is presented. The conclusion and some remarks are given in Section 3.

II. **Design of feedback controllers and main results**

We now consider the following linear time-invariant system,

$$\begin{aligned}\dot{X}(t) &= AX(t)+B_1u(t)+B_2u(t-\tau)\\ &\quad +\int_0^\tau D(s)u(t-s)ds, \end{aligned}\tag{4}$$

where $X(\cdot)\in\mathbb{R}^n$ denotes the state vector, and $u(\cdot)\in\mathbb{R}^m$ is the vector of control input. The constant matrices $A\in\mathbb{R}^{n\times n}$ and $B_1,B_2\in\mathbb{R}^{n\times m}$ are known. The lumped delay appears in the term $B_2u(t-\tau)$, while the distributed delay is represented by the integral term $\int_0^\tau D(s)u(t-s)ds$. The known function $D(\cdot):[0,\tau]\to\mathbb{R}^{n\times m}$ is assumed to be continuous, and the positive constant τ denotes the known time delay.

It is to be noticed that $X(t)$ is only the state of lumped portion of the delay system. The complete state of the system (4) at time t is $\{X(t);u(s),t-\tau\le s<t\}$. Then the initial conditions are assumed to be

$$X(0)=X_0,\quad u(t)=u_0(t),\quad \forall t\in[-\tau,0].$$

In order to design a feedback controller which stabilize the system (4), we take the following procedure.

Firstly, inspired by the reduction method in [2], [4], we introduce an integral transformation as follows,

$$W(t)=X(t)+\int_{t-\tau}^t P(t-s)u(s)ds,\tag{5}$$

where the transformation kernel matrix $P(\cdot):[0,\tau]\to\mathbb{R}^{n\times m}$ is to be determined later. Now one can calculate easily that

$$\begin{aligned}\dot{W}(t) &= \int_0^\tau(\dot{P}(s)-AP(s)+D(s))u(t-s)ds\\ &\quad +(B_1+P(0))u(t)+(B_2-P(\tau))u(t-\tau)\\ &\quad +AW(t)\end{aligned}$$

Choosing such a transformation kernel matrix $P(\cdot)$ that

$$\begin{cases}\dot{P}(s)-AP(s)+D(s)=0,\ \ s\in[0,\tau],\\ P(0)=-B_1,\end{cases}\tag{6}$$

we then get that

$$\dot{W}(t)=AW(t)+(B_2-P(\tau))u(t-\tau).\tag{7}$$

Obviously, the solution of the Cauchy problem (6) reads

$$P(s)=-e^{As}B_1-\int_0^s e^{A(s-\xi)}D(\xi)d\xi,\quad \forall s\in[0,\tau],$$

which combined with (7) yields

$$\begin{aligned}\dot{W}(t) &= \quad B_2+e^{A\tau}B_1+\int_0^\tau e^{A(\tau-s)}(s)ds\quad u(t-\tau)\\ &\quad +AW(t).\end{aligned}\tag{8}$$

Hence the integral transformation (5) is

$$\begin{aligned}W(t) &= X(t)-\int_{t-\tau}^t e^{A(t-s)}B_1u(s)ds\\ &\quad -\int_{t-\tau}^t\int_0^{t-s} e^{A(t-s-\xi)}D(\xi)u(s)d\xi ds.\end{aligned}\tag{9}$$

For the sake of simplicity, we define the matrix B as

$$B=B_2+e^{A\tau}B_1+\int_0^\tau e^{A(\tau-s)}D(s)ds,\tag{10}$$

which depends on the constant time delay τ. We then get the following linear system with lumped time-delay,

$$\dot{W}(t)=AW(t)+Bu(t-\tau).\tag{11}$$

In one word, we have reduced the system (4) with distributed input delay into the system (11) with only the lumped input delay through the transformation (9).

Secondly, we can go a step further to rewrite the system (11) as

$$\begin{cases}\dot{W}(t)=AW(t)+Bv(0,t),\\ v_t(x,t)=v_x(x,t),\ x\in(0,\tau),\\ v(\tau,t)=u(t),\end{cases}\tag{12}$$

where $v(0,t)=u(t-\tau)$ just gives the input in (11). In the sequel, we discuss under the hypothesis: *there exists such a matrix K that $(A+BK)$ is Hurwitz.* It has been proven in [1] that the orgin of the following system is exponentially stable ,

$$\begin{cases}\dot{W}(t)=(A+BK)W(t)+B\phi(0,t),\\ \phi_t(x,t)=\phi_x(x,t),\\ \phi(\tau,t)=0.\end{cases}\tag{13}$$

Aiming to transform the system (12) into the target system (13), introduce the so-called backstepping transformation,

$$\phi(x,t) = v(x,t) - \int_0^x k(x,\theta)v(\theta,t)d\theta - Q(x)^{\mathrm{T}}W(t). \quad (14)$$

The integral kennel $k(x,\theta)$ and $Q(x)$ is determined by the following system,

$$\begin{cases} k_x(x,t) + k_t(x,t) = 0, \quad k(x,0) = Q(x)^{\mathrm{T}}B, \\ Q'(x) = A^{\mathrm{T}}Q(x), \quad Q(0) = K^{\mathrm{T}}. \end{cases} \quad (15)$$

Please refer to [1] for the details. It is not difficult to obtain that

$$k(x,t) = Ke^{A(x-t)}B, \quad Q(x) = \left(Ke^{Ax}\right)^{\mathrm{T}}.$$

Thus the backstepping transformation (14) reads

$$\begin{aligned} \phi(x,t) \quad = \quad & -K \int_0^x e^{A(x-\theta)}Bv(\theta,t)d\theta + e^{Ax}W(t) \\ & + v(x,t) \end{aligned} \quad (16)$$

Noting $u(t) = v(\tau,t)$ in (12) and $\phi(\tau,t) = 0$ in (13), we then set $x = \tau$ in (16) to get

$$u(t) = K \quad e^{A\tau}W(t) + \int_{t-\tau}^t e^{A(t-s)}Bu(s)ds \quad . \quad (17)$$

Hence we actually obtain the feedback control for the original system (4),

$$\begin{aligned} u(t) \quad = \quad & K\int_{t-\tau}^t \int_{t-s}^{\tau} e^{A(t+\tau-s-\xi)}D(\xi)u(s)d\xi ds \\ & + K \quad e^{A\tau}X(t) + \int_{t-\tau}^t e^{A(t-s)}B_2u(s)ds \end{aligned} \quad (18)$$

Now we are ready to state the main result on the stabilization for the system (4).

Theorem 1: Let K be such a matrix that $A + B_2K + e^{A\tau}B_1K + \int_0^\tau e^{A(\tau-s)}D(s)Kds$ is Hurwitz. The linear time-invariant system (4) with distributed input delay can be stabilized by the feedback control $u(t)$ given in (18).

Proof We would like to prove that

$$\exists c_1, c_2 > 0, \quad \text{s. t.} \quad |X(t)| + \int_{t-\tau}^t |u(s)|ds \le c_1e^{-c_2t}. \quad (19)$$

In the sequel, c_1, c_2 stand for generic positive constants, whose values may change from line to line. In view of the form of $u(t)$ given in (17), let $u(t) = KZ(t)$ with

$$Z(t) = e^{A\tau}W(t) + \int_{t-\tau}^t e^{A(t-s)}Bu(s)ds. \quad (20)$$

It is easy to check that $Z(t)$ satisfies

$$\dot{Z}(t) = (A + BK)Z(t).$$

By the assumption on K, it is known that

$$|Z(t)| \le c_1e^{-c_2t}$$

for some positive constants c_1, c_2. So we can conclude after some simple calculations that

$$\int_{t-\tau}^t |u(s)|ds \le c_1e^{-c_2t}.$$

Naturally, $|W(t)|$ decays exponentially in view of (20), which together with (9) implies $|X(t)| \le c_1e^{-c_2t}$. This completes the proof.

III. CONCLUSIONS

In this paper, we present a feedback stabilization control law for linear time-invariant systems with lumped and distributed input delay. The system under consideration in this paper is a general model for linear time-invariant systems with delay in the control input. The result obtained in this paper can cover the stabilization result for linear systems with lumped input delay only in [4].

In fact, the system (4) degenerates to the system (1) if $D(\cdot) \equiv 0$. And the feedback control $u(t)$ given in (18) now reads

$$u(t) = K \quad e^{A\tau}X(t) + \int_{t-\tau}^t e^{A(t-s)}B_2u(s)ds \quad , \quad (21)$$

which seems to differ by a factor $e^{A\tau}$ from (3) derived in [4]. If examining (20), we then find out that (20) is a new transformation which reduce the time-delayed system (11) into the delay-free system

$$\dot{Z}(t) = AZ(t) + Bu(t).$$

This transformation differs also by a factor $e^{A\tau}$ from the reduction transformation (2), which coincides with the difference between (21) and (3). This relation was discovered too in [7] through discussing the relations between continuous reduction transformation and discrete one.

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