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A Diagrammatic Classification in a Combinatorial Problem: The Case of a Stable Marriage Problem

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Abstract: The Stable Marriage Problem (SMP) is a combinatorial problem to find stable matching between n women and n men given a complete preference list of men over women and vice versa. An instance of SMP can be expressed by a bipartite graph with multiple (weighted) edges. By rearranging the graph, we use a diagram that involves several constraints to visualize several symmetries. By the diagram, all the instances of the size three SMP (three women and three men) are classified. The classification may be supported by the fact that the same class has the same stable matching.

Keywords: Graph equivalence, diagrammatic classification, stable marriage problem, stable matching, decompositi on, bipartite graph, stable marriage graph.

I. INTRODUCTION

After the proposal of the matching problem by Gale and Sharpley [1], many matching problems including stable marriage problem (SMP) [2] have been studied extensively. Stable marriage problem is a combinatorial problem to find stable matching between n women and n men given a complete preference list of men over women and vice versa. Stable marriage problem has several variants such as a job assignment problem, roommate problem varying the assumption on the set of members to be matched.

Although the stable marriage problem has too strong assumptions (such as complete list of preference) to be applied to practical problems, symmetries embedded in the problem keep us being attracted. Thus, we tried to visualize the problem by network visualization tool [3]. This paper further focuses the decomposability of the problem by involving diagrams such as a bipartite graph and a stable marriage graph [4, 5].

As a preliminary example, we presented a classification of indecomposable structure of size three (three men by three women) SMP. Decomposition of the problem may be a first thing to do in tackling a complex problem. For SMP, component decomposition [2] and weak (but applicable to the preference matrix) decomposition [6] have been studied. In classifying the size three SMP, we tried to generalize a concept of *mutual infatuation* (a pair of man and woman who rank the partner first). A cycle of fist rank relation plays an important role in classifying the SMP.

Section II presents the stable marriage problem and its graphical representation. Section III defines equivalence of instances of SMP through graph homomorphism (equivalence). Section III presents a classification of SMP by diagrams (bipartite graph and marriage graph).

II. STABLE MARRIAGE PROBLEM

The Stable Marriage Problem (SMP) assumes n women and n men each of them has an ordered preference list (or a ranking) without tie to the opposite sex. As in the example shown in Fig. 1, the men m_2 has an ordered preference list (w_3, w_2, w_1) or a ranking (3, 2, 1), which means m_2 likes w_3 best, and he prefers w_3 to w_2, w_2 to w_1. That is, there is an injection (one to one, but not necessarily onto) mapping from a set of women (men) to an element of permutation group of size n such as shown in the ranking by each person (Fig. 1). We will use a graph that extracts a specific rank (such as the first preference) in classification.

Under the above assumptions, SMP seeks for the complete matching between n women and n men (a bijection from n women to n men), which satisfies stability. The stability requires the concept of blocking pair. Two pairs (m_i, w_p) and (m_j, w_q) are blocked by the pair (m_i, w_q) if m_i, prefers w_q to w_p and w_q prefers m_i to m_j. For example, a pair (m_2, w_3) and (m_3, w_2) will be blocked by the pair (m_2,w_2). A complete matching without being blocked is called *stable* matching.

An instance of SMP can be expressed by a multiple (arcs) bipartite graph in a straightforward way such that an arc of *k*-th order from m_i, to w_j is directed if mi ranks w_j j *k*-th. Another graphical expression of an instance is the *stable marriage graph* [4,5]. For SMP with size n by n, n^2 nodes of possible pairs (m_i,,w_j) are placed in a matrix form. For a row of m_i,, an arc from (m_i,,w_p) to (m_j, w_q) is placed if m_i, prefers w_q to w_p. Fig. 2 shows a stable marriage graph corresponding to the instance shown in Fig. 1. A stable marriage graph can be simplified (as in Fig. 2). In a simplified stable marriage graph, redundant nodes may be abbreviated if it can be derived by a transitivity. For example, if m_1 prefers w_1 to w_2, and w_2 to w_3, it follows that m_1 prefers w_1 to w_3 hence the corresponding arc is abbreviated.

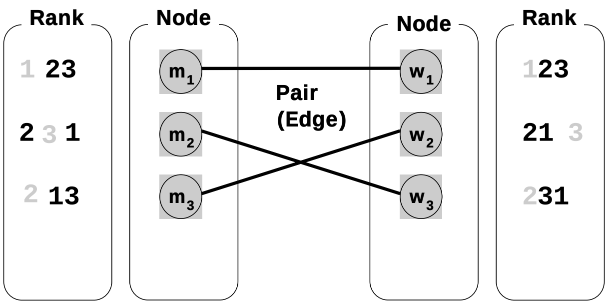


Fig. 1. An illustration of Stable Marriage Problem with size three.

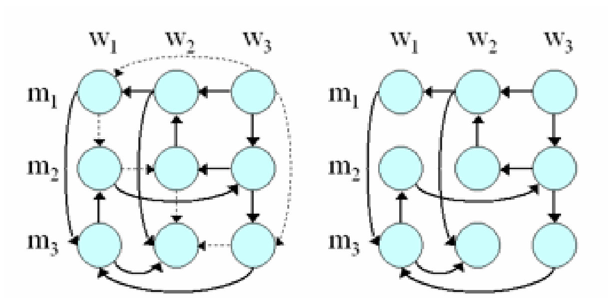


Fig. 2. A stable marriage graph (left) and its simplified one by removing redundant arcs with transitivity (right).

III. EQUIVALENCE OF INSTANCES THRO UGH GRAPH

Prime numbers remain to be a mystery not only in discrete mathematics such as Integer Theory but in continuous mathematics such as theory of functions. Prime numbers also plays an important role of an engine for mathematics and other applied mathematics such as cryptography.

Prime numbers are defined on the set of integers by means of factorization with division operation. The prime ness could be defined on other mathematical objects with structures such as instances of Stable Marriage Problem. With expectation of the important role played by "primeness" in the instances of SMP, we study several type of decomposition in instances of SMP, and indecomposable structure of them.

Decomposition of SMP had been traced back to the book by Gusfield and Irving [2], and have been studied [6]. Here, we focus on a decomposability that will be naturally considered by two independence concepts: the concept of extended individual and an independent pair. The extended individual is a set of individuals that can be dealt as if one individual in finding stable matching. The independent stable pair is the two entities that can be paired without affecting any other possible pairs, and hence the pair appears in any stable matching. The decomposition leads to indecomposable structures, yielding a full classification of SMP with size three.

As an example of extended individuals, consider the case of SMP with nationality in members. Half of the members are Japanese and the other half American both in female and male. Assume any woman (man) prefers man (woman) with the same nationality to man (woman) otherwise. Then Japanese women (men) as well as American women (men) can be treated as if they are one woman (man). We use this example also as an example of trivial decomposition of SMP. In fact, this SMP is actually two independent SMPs: one between Japanese women and men and another between American women and men.

As a trivial example of independent pair, woman and man who mutually rank first can be paired without affecting any other paring. A nontrivial example would be the pair who would turn out to be mutual first rank only after removing the above the trivial independent pair. We call this latter one the independent pair hidden by the former obvious independent pair.

IV. CLASSIFICATION VIA DIAGRAMS

1. Indecomposable Structure by Bipartite Graph

Trivial example of the indecomposable structure in SMP is those with size one (Fig. 3). Obvious example may be the size two (Fig. 4), for we need to eliminate the graph of the size one (otherwise it can be decomposed to two size one SMPs). In the enumeration, we do not distinguish the ones that can be mapped by the label exchange within the same sex, and exchange the sex with all the members.

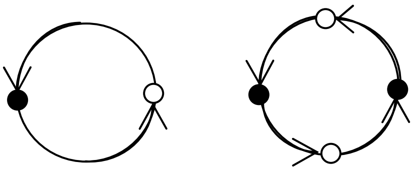


Fig.3. Indecomposable structure of size one (left) and size two (right). Nodes with diffe rent color indicate persons with different sex, and arcs indicate the fist preference.

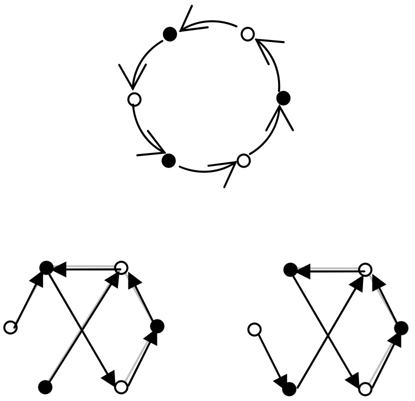


Fig. 4. Indecomposable structure of size three SMP. Arcs of the fist preferences are shown. They will be called type 0 (above), 1 (below, left) and 2 (below, right)

2. Indecomposable Structure by Stable Marriage Graph

Stable marriage graph can be reduced by removing nodes dominated by a man optimal node or a woman optimal node, since the pairs corresponding to the nodes dominated do not appear in any stable matching.

By a graph homomorphism of reduced stable marriage graph, all the instances of SMP with size three can be classified to the six classes (Fig. 5). Among them, classes 3, 4, 5 and 6 correspond to indecomposable structure.

We will further classify the SMP with size three by a multiple (arcs) bipartite graph, since several operations are required to obtain the reduced stable marriage graphs, although once obtained they even reveal the structure of stable matching.

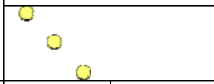
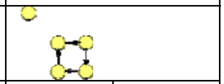
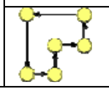
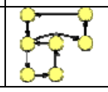
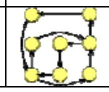
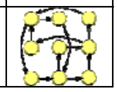
Decomposable size 3 SMP	class 1		class 2	
				
Indecomposable size 3 SMP	class3	class4	class5	class6
				

Fig.5. Classification of size 3 SMP by a graph homomorphism in the reduced stable marriage graph.

3. Classification of Indecomposable SMP

We will further classify three types based on the bipartite graph of second preference.

Type 0 (Fig. 6) is the simplest, since they can be mapped to class 3, 4, 5, and 6, respectively, by the number of cycles of length two in the bipartite graph of second preference (Fig. 6).

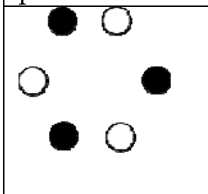
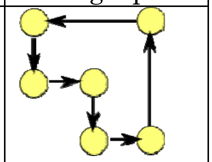
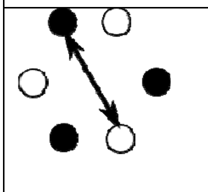
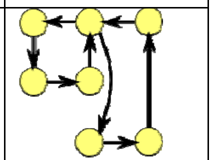
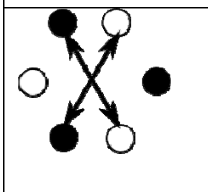
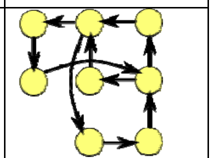
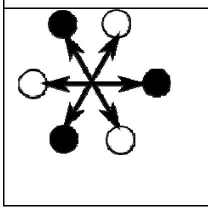
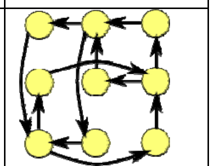
Cycles of 2nd pref.	Reduced SMP graph
	 class 3
	 class 4
	 class 5
	 class 6

Fig. 6. Indecomposable structures of type 0, size three SMP.

Type 1 (Fig. 7) can be divided into five subtypes, one of which belongs to class 4 (Fig. 5) and the rest to class 3. In the graph, the arcs (of second preference) included in a cycles of length four that appears in the first preference (Fig. 4) but opposite directions are omitted. It can be observed that when the class is upgraded from class 3 to 4, the arc is added and the number of cycles with length two increases. The graph obtained by adding an arc to the original graph is placed to the right of the original graph.

Type 2 (Fig. 8) are divided into twenty subtypes. Among them, twelve subtypes belong to class 3, seven to class 4, and one to class 5. Again, it can be observed that when the class is upgraded, the arc is added. The number of cycles with length two increases when the class is upgraded from 3 to 4 and 4 to 5, however, it is not true from 3 to 5, although the length 4 cycle with the first preference and second preference mixed appears in the jump from the class 3 to 5.

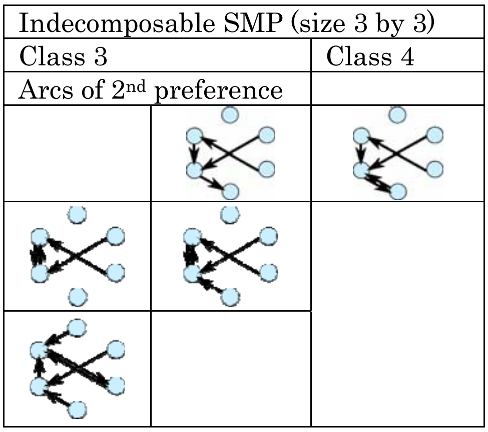


Fig. 7. Indecomposable structures of type 1, size three SMP.

V. SUMMARY

We first classified all the instances of the size three stable marriage problem (SMP) into six classes by the reduced stable marriage graph. Among six classes, four classes correspond to indecomposable structure. Then, we have classified all the instances of indecomposable stable marriage problem with size three into three types based only on the bipartite graph of the first preference. Each type can further be classified to subtypes based on the second preference structure. By mapping these structure to the three classes induced by the reduced stable marriage graph, it is shown that the cycle in a bipartite graph plays a curtail role.

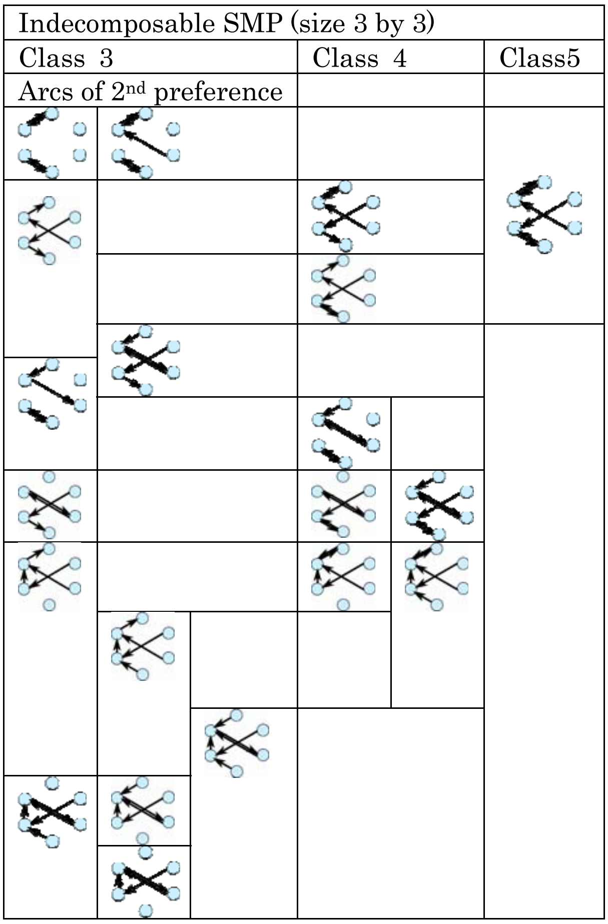


Fig. 8. Indecomposable structures of type 2, size three SMP.

REFERENCES

[1] Gale D and Shapley LS (1962), College admissions and the stability of marriage, *American Mathematical Monthly*, 69:9-15
[2] Gusfield D and Irving RW (1989), *The Stable Marriage Problem: Structure and Algorithm*, MIT Press, London
[3] Morizumi Y, Hayashi T, Ishida Y. (2009) A Network Visualization of Stable Matching in Stable Marriage Problem, *Proc of AROB*
[4] M. Balinski and G. Ratier, Of stable marriages and graphs, and strategy and polytopes. *SIAM Review* 39 (1997) 575 – 604.
[5] Nemoto T. (2002) *Handbook of Applied Mathematical Programming*, ASAKURA SYOTEN, 779-830 (in Japanese)
[6] Veklerov E (1989), On the decomposability of the stable marriage problem. *BIT Numerical Mathematics*, 29(1): 41-46