A most simple, non-computer-aided proof of the Four-Color Theorem

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Abstract: An alternative brief proof of the Four Color Theorem without using a computer is described. The proof is essentially similar to the brief proof very recently described by the author as a bird's-eye view, but is much more simple than that, and more importantly, differs from that in the way of vertex-reducing of complete triangulation graph. This new, most simple proof is much shorter than the recently described one, and both of these two proofs are far more easily understood than Appel and Haken's proof in 1977. These new findings of non-computer-aided proofs clearly tell us about what the essential portion of the enormous complexity of the Four Colour Theorem is.
Keywords: Triangulation, Vertex reduction of colored triangulation graph

I. INTRODUCTION

The Four-Colour Conjecture (FCC) concerning map-coloring first proposed by Guthrie [1], had long been one of the most difficult unsolved problems in mathematics. In 1977, Appel and Haken (A&H) [2] succeeded in proving the FCC using a computer. The question whether or not the Four-Colour Theorem (FCT) could be proved without using a computer has since arisen as a most important problem remaining to be solved. Very recently, I have presented the details of a bird's-eye view of a very plausible brief proof of the FCT without using a computer [3]. In this paper, an alternative brief proof is described which is much more simple, although it is essentially similar to the brief proof described in Ohnishi (2009) [3], but considerably differs from that in the way of vertex-reducing of complete triangulation graph on sphere, S^2.

II. PRELIMINARIES

In this section, some basic definitions useful for achieving a most simple proof of the FCT are described. "▌" denotes the end of each definition or theorem, whereas "█" denotes the end of each proof.

The following terms are defined as given in the previous paper [3].

Jordan curve QQ' and its *endpoints Q* and *Q':* see Definition 1 in ref. [3].

Internal and *external domains* (*int C, ext C, Int C, Ext C,* where *C* is a closed Jordan curve.): see Theorem 1 in ref.[3].

Graph, vertex, spherical graph $G(S^2)$, connected graph: see Definitions 2 and 5 in ref.[3].

Valency (degree) of a vertex P, written as *val P :* see Definition 3 in ref.[3].

ν-colorable; ν-colored graph written as $col^{\nu}(G)$: see Definitions 8 in ref.[3].

$col^{\nu}(P)= a$, (where $P \in G$), as denoting that vertex P is colored with a in $col^{\nu}(G)$: see Definitions 8 in ref.1.

Kempe-block, ab-Kempe-blocks written as $K_{ab}(P_i)$, $K_{ab}(P_i, P_j)$ in $col^4(G)$: see Definition 10 in ref.[3].

Definition 1: <u>*edge, adjacent:*</u> *Edge* is defined as a Jordan curve connecting and *ex*cluding two vertices (which are end-points) *P* and *P'*. An edge *e*, connecting two vertices *P* and *P'* is written as $e = [P,P']$. A vertex *P* is called to be *adjacent to P'*, if a graph Γ has an edge, $[P,P']$. ▌

Note that **Definition 1** differs from the definition of "edge" given in the previous paper (see **Definition 2** in ref.[3]), and is identical to the definition of Berge [4]. Accordingly, some terms need to be re-defined based on Definition 1.

The next theorem is well-known as given in Ore [5].

Theorem 1: If *C* is a closed Jordan curve on a sphere S^2, then we have $S^2 = int\ C + ext\ C + C = Int\ C + Ext\ C - C$. ▌

Definition 3: <u>*s-cycle, s-gon, s-path:*</u> An (*s*-)cycle is defined by a *s*-vertex-graph, $C = C^s = C^s(e_{12},e_{23},\ldots,e_{s,1}) = P_1 + e_{12} + P_2 + e_{23} + \ldots + e_{s-1,s} + P_s + e_{s,1}$. An (*s*-)path is defined by $U(P_1,P_s) = U^s(P_1,P_s) = C^s(e_{12},e_{23},\ldots,e_{s,1}) - e_{s,1}$. C^s is also called *s*-gon (= *s*-hedron) (e.g., poly-gon, di-gon, tri-angle, tetrahedron=quadrilateral, pentagon, etc.). ▌

Definition 4: *face:* If $G(S^2)$ has a *s*-cycle (= *s*-gon), C^s, where $int\ C^s = ø$, then $int\ C^s$ is called *face* (or *s-gon face*). ▌

Thus we find $S^2 = int\ C^s + C^s + ext\ C^s$.

Definition 5: *(complete) triangulation:* If $G(S^2)$ is a connected graph dividing S^2 into exclusively triangular faces, G is called *"complete triangulation (of S^2)"*. If "$G(S^2) = C^s$" satisfies $ext\ C^s = ø$, and if G divides $Int\ C^s$ into exclusively triangular faces, G is called *"triangulation of s-gon, C^s"*. ■

The next theorem is well-known [6,7,11], as described in [3].

Theorem 2: Let $T(S^2)$ be an arbitrarily selected complete triangulation of S^2. The four-color theorem (FCT) is equivalent to the statement that "*Proposition A* is true", where *Proposition A* is given by;

Proposition A: $T(S^2)$ is vertex four-colorable. ■

Definition 6: *Two-faced quadrilateral (Fig. 1):* "*Two-faced quadrilateral* with a diagonal edge e_{13}" is defined as a subgraph of $G(S^2)$, and is given by $Q^{2f} = C^4{}_0 + e_{13} \subseteq G$, where $C^4{}_0 = C^4(e_{12},e_{23}, e_{34}, e_{41})$, $e_{ij} = [P_i, P_j]$, $e_{13} \subset int\ C^4{}_0$, and e_{13} is a boundary edge dividing $int\ C^4$ into two triangular faces. Q^{2f} is written as $Q^{2f} = Q^{2f}(C^4{}_0;e_{13})$. If Q^{2f} in $G(S^2)$ has any edge, e'_{13} or e'_{24}, satisfying $e'_{13} = [P_1,P_3] \subset ext\ C^4{}_0$, or $e'_{24} = [P_2,P_4] \subset ext\ C^4{}_0$, then the Q^{2f} is called *"incomplete quadrilateral"*, whereas it is called *"complete quadrilateral"* if there is none of such edges. *$G(S^2)$ having its sub-graph $Q^{2f}{}_0 = Q^{2f}(C^4{}_0;e_{13})$* is written as $G = G(Q^{2f}{}_0;C^4{}_0)$. ■

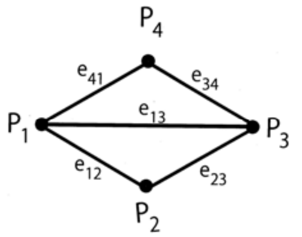

Fig.1 Two-faced quadrilateral, $Q^{2f}(C^4{}_0;e_{13})$, where $C^4{}_0$ is a 4-cycle (= quadrilateral) given by $C^4{}_0 = C^4(e_{12},e_{23},e_{34},e_{41})$, $e_{ij} = [P_i,P_j]$, $e_{13} \subset int\ C^4{}_0$. $\{Q^{2f}\}$ is an unavoidable (one-element-)set of subgraphs in $T_k(S^2)$, a complete triangulation of S^2 with$k$ vertices ($k \geq 4$). see Definition 6.

III. BASIC THEOREMS

The following basic theorems are useful for proving the FCT.

Theorem 3: For *an ab*-Kempe-block, $K_{ab}(P_i, P_j) \in G(S^2)$, where P_i and $P_j \in G(S^2)$, there exists a 2-coloured path $U_{ab}(P_j,P_j)$ as a subgraph of the 2-coloured graph, $K_{ab}(P_i, P_j)$. ■

[Proof] Evident from the definitions of connected graph (Definition 5 in ref.[3].) and vertex 2-colored graph (Definition 8 in ref.[3].). ■

This theorem means that P_i and P_j are connected by a *2-coloured Jordan curve*, $U_{ab}(P_j,P_j)$, which is a 2-coloured path.

Theorem 4: Let $T_k(S^2)$ be a complete triangulation of S^2, having k vertices ($k \geq 4$). Then there exists at least one quadrilateral given by $Q^{2f}{}_{k,0} = Q^{2f}(C^4{}_{k,0};e_{13}) \subset T_k$, where $C^4{}_{k,0} = C^4(e_{12},e_{23},e_{34},e_{41})$, $e_{13} \subset int\ C^4{}_{k,0}$, and $e_{ij} = [P_i, P_j]$. Furthermore, $Q^{2f}{}_{k,0}$ satisfies $val\ P_1 \geq 3$, $val\ P_3 \geq 3$, $val\ P_2 \geq 2$, and $val\ P_4 \geq 2$. ■

[Proof] Let e_{13} (= $[P_1,P_3]$) be an arbitrarily selected edge of $T_k(S^2)$, then we find two cases (i) and (ii), as below;

(i) In the case where one or more 2-cycles exist as subgraph(s) of $T_k\ (S^2)$ ($k \geqq 4$ as shown in Fig. 2[A]. Let an arbitrarily selected 2-cycle be $C^2{}_k(e_{13}, e'_{31})$, where $eij = [P_i,P_j]$, then one finds that the edge e_{13} is a boundary of two 3-gon (triangular) faces, whose boundaries are 3-cycles, $C^3(e_{12}, e_{23}, e_{31})$ and $C^3(e_{13}, e_{34}, e_{31})$ (where $e_{ij} = [P_i,P_j]$.), as shown in Fig.1 [A]. Thus there exist a quadrilateral, $G = G(Q^{2f}{}_0;C^4{}_0)$, where $C^4{}_0 = C^4(e_{12}, e_{23}, e_{34}, e_{41})$, and e_{13} is a diagonal edge. Since $C^2{}_k(e_{13}, e'_{31})$ is a closed Jordan curve, there does not exist $e_{34} = [P_3,P_4]$. Then it follows that $val\ P_1 \geqq 4$, $val\ P_3 \geqq 4$, $val\ P_2 \geqq 2$, $val\ P_4 \geqq 2$.

(ii) In the case where any 2-cycle does not exist as subgraph of $T_k(S^2)$. By letting an arbitrarily selected edge be denoted by $e_{13} = [P_1,P_3]$, P_2 and P_4 are found to be different two vertices (See Fig 2[B]), since an identical single vertex of P_2 and P_4 means the existence of a 2-cycle, $C_2(e_{12}, e_{21})$, which is in conflict with the case (ii), and belongs to the case (i). In case (ii), we thus find a quadrilateral, $G = G(Q^{2f}{}_0;C^4{}_0)$, where $C^4{}_0 = C^4(e_{12}, e_{23}, e_{34}, e_{41})$, and e_{13} is a diagonal edge, as shown in Fig. 2[B], satisfying $val\ P_1 \geqq 3$, $val\ P_3 \geqq 3$, $val\ P_2 \geqq 3$, $val\ P_4 \geqq 3$.

Thus we find that there exist a quadrilateral, $G = G(Q^{2f}{}_0;C^4{}_0)$, where $C^4{}_0 = C^4(e_{12}, e_{23}, e_{34}, e_{41})$, and e_{13} is a diag onal edge, and $val\ P_1 \geqq 3$, $val\ P_3 \geqq 3$, $val\ P_2 \geqq 2$, $val\ P_4 \geqq 2$. ■

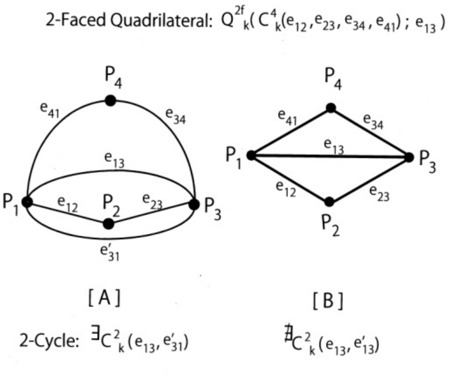

Fig. 2. Proof of Theorem 3.1 showing the existence of two-faced quadrilateral in $T_k(S^2)$ ($k \geqq 4$)

[A] The case (i) where a 2-cycle, $C^2{}_k(e_{13}, e'_{31})$, exists as a subgraph of $T_k\ (S^2)$ ($k \geqq 4$). ($val\ P_1 \geqq 4$, $val\ P_3 \geqq 4$, $val\ P_2 \geqq 2$, $val\ P_4 \geqq 2$). [B] The case where any 2-cycle does not exist as a subgraph of $T_k\ (S^2)$ ($k \geqq 4$), ($val\ P_1 \geqq 3$, $val\ P_3 \geqq 3$, $val\ P_2 \geqq 3$, $val\ P_4 \geqq 3$).

Lemma 4.1: In Theorem 4, a set, $\{Q^{2f}{}_k\}$ is an unavoidable set (See [6,8] for definition.) of sub-graphs in $T_k(S^2)$, and consists of only one element being a quadrilateral. ■

[Proof] Evident from Theorem 4. ■

Theorem 5 : For an arbitrarily selected four-colorable complete triangulation graph with k vertices, $T_k(S^2)$ ($k \geq 4$), let $col^4{}_0(T_k)$ be *a* 4-coloured graph of T_k. Concerning an arbitrarily selected quadrilateral $Q^{2f}{}_{k,0} = Q^{2f}(C^4{}_{k,0}\ ;\ e^{(k)}{}_{13})$ $(\subset T_k)$, where $C^4{}_{k,0} = C^4(P_1{}^{(k)},P_2{}^{(k)},P_3{}^{(k)},P_4{}^{(k)})$ and $e^{(k)}{}_{13} = [P_1{}^{(k)},P_3{}^{(k)}] \subset int\ C^4{}_{k,0}$, $col^4{}_0(T_k)$ belongs to either one of the two types, type I and type II, defined by:

type I: $K_{ac}(P_1{}^{(k)},P_3{}^{(k)})$ $(\subset col^4(T_k;\ Q^{2f}{}_0))$ exists.

type II: $K_{ac}(P_1{}^{(k)},P_3{}^{(k)})$ $(\subset col^4(T_k;\ Q^{2f}{}_0))$ does not exist. ■

[Proof] Evident. ■

Definition 6 : (case I and case II 4-colorations) Let $col^4{}_{\mathrm{I}}(T_k;Q^{2f}{}_0)$ and $col^4{}_{\mathrm{II}}(T_k;Q^{2f}{}_0)$ denote 4-colored complete triangulation graph of case I and that of case II (in Theorem 5), respectively. ■

III. FINAL PROOF

Definition 7 (Vertex-reducing operations f_1 and f_2): Definitions, f_1 and f_2, (and there reverse operations $f_1{}^{-1}$ and $f_2{}^{-1}$), are given by Definition 12 in ref.[3]. Suboperations f_{1a}, f_{1b} and f_{2a} f_{2b} satisfying $f_1 = f_{1b}(f_{1a})$, and $f_2 = f_{2b}(f_{2a})$ are defined as schematized in Fig.3 and Fig.4 ■

Theorem 6. Let a complete triangulation $T_k(S^2)$ be four-colorable ($k \geq 4$), then $T_{k-1}(S^2)$ *defined by* $T_{k-1} = T_{k-1}(T_k;\ Q^{2f}{}_{k,0}) = f(T_k;\ Q^{2f}{}_0)$ is four-colorable, where $Q^{2f}{}_{k,0} = Q^{2f}(C^4{}_{k,0}\ ;\ e^{(k)}{}_{13}) \subset T_k$, $C^4{}_{k,0} = C^4(P_1{}^{(k)},P_2{}^{(k)},P_3{}^{(k)},P_4{}^{(k)})$, and $e^{(k)}{}_{13} = [P_1{}^{(k)},P_3{}^{(k)}] \subset int\ C^4{}_{k,0}$, and further, f denotes a operation given by $f = f_1$ or f_2 for type I or type II 4-coloration of $T_k(Q^{2f}{}_0, e^{(k)}{}_{13})$, respectively. Then $T_{k-1}(T_k;\ Q^{2f}{}_{k,0})$ is 4-colourable. ■

[Proof] Let $col^4(T_k;\ Q^{2f}{}_0)$ be a 4-coloured graph of $T_k(S^2)$. Vertex-recucing operations of $col^4(T_k;\ Q^{2f}{}_0)$ are as below;

Case I: If $col^4(T_k;\ Q^{2f}{}_0)$ is type I in Theorem 5, then operation of graph modification $f_1(T_k;\ Q^{2f}{}_0) = f_{1b}(f_{1a}(T_k;\ Q^{2f}{}_0))$ gives a reduced triangulation graph $T_{k-1}(T_k;\ Q^{2f}{}_{k,0})$ *and its 4-coloured graph,* $col^4{}_I(T_{k-1};\ Q^{2f}{}_{k,0})$, as shown in Fig.3. Note that $T_{k-1}(T_k;\ Q^{2f}{}_{k,0})$ and $col^4{}_{\mathrm{I}}(T_{k-1};\ Q^{2f}{}_{k,0})$ are defined by $T_k(S^2)$ and $col^4(T_k;\ Q^{2f}{}_0)$, respectively. Thus $T_{k-1} = f_1(T_k;\ Q^{2f}{}_0)$ is 4-colourable.

Case II: If $col^4(T_k;\ Q^{2f}{}_0)$ is type II in Theorem 5, then operation of graph modification $f_2(T_k;\ Q^{2f}{}_0) = f_{2b}(f_{2a}(T_k;\ Q^{2f}{}_0))$ gives a reduced triangulation graph $T_{k-1}(T_k;\ Q^{2f}{}_{k,0})$ *and its 4-coloured graph,* $col^4{}_{II}(T_{k-1};\ Q^{2f}{}_{k,0})$, as shown in Fig.4. Note that $T_{k-1}(T_k;\ Q^{2f}{}_{k,0})$ and $col^4{}_{\mathrm{I}}(T_{k-1};\ Q^{2f}{}_{k,0})$ are defined by $T_k(S^2)$ and $col^4(T_k;\ Q^{2f}{}_0)$, respectively. Thus $T_{k-1} = f_1(T_k;\ Q^{2f}{}_0)$ is 4-colourable.

Acordingly, $T_{k-1}(T_k;\ Q^{2f}{}_{k,0})$ is 4-colourable. ■

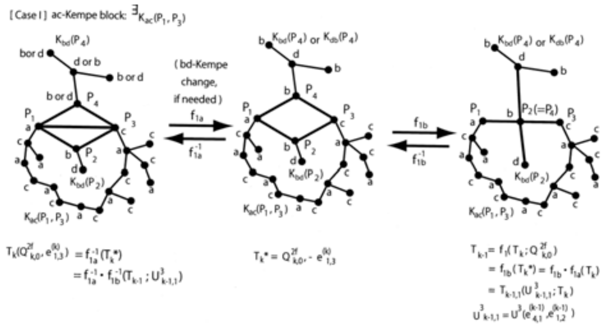

Figure 3. Reversible vertex-reduction and its reverse operation of a 4-coloured complete triangulation graph, $col^4{}_0(T_k\ ;\ Q^{2f}{}_{k,0})$ in case I, where there exists an *ac*-Kempe block, $K_{ac}(P_1, P_3)$, in $Ext\ C^4{}_{k,0}$. The left and right schematized $col^4{}_{0,I}(T_k;\ Q^{2f}{}_0)$ and $col^4{}_{0,I}(T_{k-1};\ Q^{2f}{}_{k,0})$, respectively. (From Fig. 4 in [3])

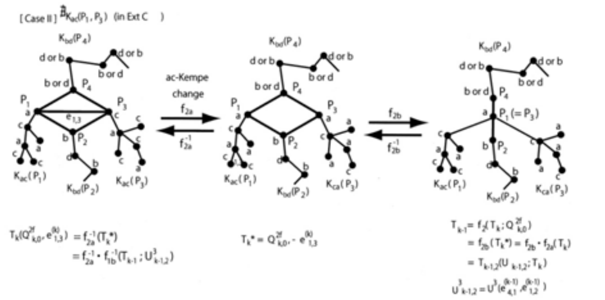

Figure 4. Reversible vertexreduction and its reverse ope ration of a 4coloured complete triangulation graph, $col^4{}_0(T_k\ ;\ Q^{2f}{}_{k,0})$ in case II, where there does not exist any *ac*Kempe block, $K_{ac}(P_1,P_3)$, in $Ext\ C^4{}_{k,0}$. The left and right schematized $col^4{}_{0,II}(T_k;\ Q^{2f}{}_0)$ and $col^4{}_{0,II}(T_{k-1};\ Q^{2f}{}_{k,0})$, respectively. (From Fig.5 in [3])

Theorem 7. Let $T_k(S^2)$ be 4-colourable, then there exists a series of 4-coloured complete triangulation graphs, $\{ col^4{}_0(T_k\ ;\ Q^{2f}{}_{k,0}),\ col^4{}_0(T_{k-1};\ Q^{2f}{}_{k,0}),\ \ldots\ ,\ col^4{}_0(T_4\ ;\ Q^{2f}{}_{5,0}),\ col^4{}_0(T_3\ ;\ Q^{2f}{}_{4,0}) \}$, where $col^4{}_0(T_{k-i}\ ;\ Q^{2f}{}_{k-i+1,\ 0})$ ($i=1,2,\ \ldots\ ,\ k-3$) is defined by $T_{k-i} = f(T_{k-i+1};\ Q^{2f}{}_{k-i,0})$, where $f = f_{\mathrm{I}}$ (for type I coloration of T_{k-i+1}) or f_{II} (for type II coloration of T_{k-i+1}), and $Q^{2f}{}_{k-i+1,0}$ is an arbitrarily selected quadrilateral of T_{k-i+1}. ■

[Proof] From theorm 6, $col^4{}_0(T_{k-i}\ ;\ Q^{2f}{}_{k-i+1,0}) = f(col^4{}_0(T_{k-i+1}\ ;\ Q^{2f}{}_{k-i+1,0}))$. Repeated applications of this equation to $T_k(S^2)$, for $i = 1,2,\ \ldots,\ k-3$, we obtain a series of 4-coloured complete triangulations, $\{ col^4{}_0(T_k\ ;\ Q^{2f}{}_{k,0}),\ col^4{}_0(T_{k-1}\ ;\ Q^{2f}{}_{k,0}),\ \ldots\ ,\ col^4{}_0(T_4\ ;\ Q^{2f}{}_{5,0}),\ col^4{}_0(T_3\ ;\ Q^{2f}{}_{4,0}) \}$. Note that $col^4{}_0(T_3\ ;\ Q^{2f}{}_{4,0})$ is a 4-coloured 3-gon graph (3-vertex complete graph) whose coloration depends on $col^4{}_0(T_k\ ;\ Q^{2f}{}_{k,0})$. ■

Lemma 7.1: The necessary condition for a complete The necessary condition for a complete triang ulation graph, $T_k(S^2)$, ($k \geq 4$), to be four colorable is the e xistence of $col^4{}_0(T_3\ ;\ Q^{2f}{}_{4,0})$, defined in Theorem 7. ■

[Proof] Evident from theorem 7. ■

Lemma 7.2: The necessary condition for a complete triangulation graph, $T_k(S^2)$, *(k≥4),* to be four colorable, defined in Lemma 7.1 is satisfied. ∎

[Proof] $col^4_0(T_3 ; Q^{2f}_{4,0})$ is a 4-coloured 3-gon graph (3-vertex complete graph) whose coloration depends on $col^4_0(T_k ; Q^{2f}_{k,0})$. Therfore, $col^4_0(T_3 ; Q^{2f}_{4,0})$ really exists as one of the 4x3x2 = 24 4-colorations of the 3-gon graph. ∎

Theorem 8. For a given coloration, $col^4_0(T_3 ; Q^{2f}_{4,0})$, which is one of the 24 possible coloration of trigon, $col^4_0(T_4 ; Q^{2f}_{5,0})$, given inTheorem 7 can be reconstructed from $col^4_0(T_3 ; Q^{2f}_{4,0})$. ∎

[Proof] Since $col^4_0(T_3 ; Q^{2f}_{4,0})$ really exists from Lemma 7.1, then $col^4_0(T_4 ; Q^{2f}_{5,0})$ can be reconstructed by $col^4_0(T_4 ; Q^{2f}_{5,0}) = f^1 (col^4_0(T_3 ; Q^{2f}_{4,0}))$, because the operation f defined in Fig. 3 is reversible graph modification. Thus, by applying $col^4_0(T_4 ; Q^{2f}_{5,0}) = f^1 (col^4_0(T_3 ; Q^{2f}_{4,0}))$ for all of the possible operations of f^1, there exists at least one operation generating $col^4_0(T_4 ; Q^{2f}_{5,0})$, whether or not T_4, $Q^{2f}_{4,0,}$ and $col^4_0(T_3 ; Q^{2f}_{4,0})$ are known. ∎

Lemma 8.1. For a given coloration, $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0})$ (i=k-3, k-4, …, 2, 1), defined by Theorem 8, $col^4_0(T_{k-i+1} ; Q^{2f}_{k-i+2,0})$ given inTheorem 7 can be reconstructed from $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0})$. ∎

[Proof] Since $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0})$ really exists, from Lemma 7.2, for i = k-4, $col^4_0(T_5 ; Q^{2f}_{6,0})$ can be reconstructed by $col^4_0(T_5 ; Q^{2f}_{6,0}) = f^1 (col^4_0(T_4 ; Q^{2f}_{5,0}))$, because the operation f defined in Fig. 3 is reversible graph modification. Thus, by applying $col^4_0(T_5 ; Q^{2f}_{6,0}) = f^1 (col^4_0(T_4 ; Q^{2f}_{5,0}))$ for all of the possible operations of f^1, there exists at least one operation generating $col^4_0(T_5 ; Q^{2f}_{6,0})$, even if T_5, $Q^{2f}_{6,0,}$ and $col^4_0(T_5 ; Q^{2f}_{6,0})$ are unknown.

Similarly, for k-4 ≤ i ≤ 2, when $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0})$ can be known to exist by $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0}) = f^1 (col^4_0(T_{k-i-1} ; Q^{2f}_{k-i,0}))$, $col^4_0(T_{k-i+1} ; Q^{2f}_{k-i+2,0})$ can be reconstructed by $col^4_0(T_{k-i+1} ; Q^{2f}_{k-i+2,0}) = f^1 (col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0}))$, because the operation f defined in Fig. 3 is reversible graph modification. Thus, by applying $col^4_0(T_{k-i+1} ; Q^{2f}_{k-i+2,0}) = f^1 (col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0}))$ for all of the possible operations of f^1, there exists at least one operation generating $col^4_0(T_{k-i+1} ; Q^{2f}_{k-i+2,0})$, even if T_5, $Q^{2f}_{6,0,}$ and $col^4_0(T_5 ; Q^{2f}_{6,0})$ are unknown.

The final similar application of $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0}) = f^1 (col^4_0(T_{k-i-1} ; Q^{2f}_{k-i,0}))$, for i = 0, it follows that $col^4_0(T_k ; Q^{2f}_{k,0}) = f^1 (col^4_0(T_{k-1} ; Q^{2f}_{k,0}))$, as shown in Fig.4 and Fig.5. Thus the initial coloration of $col^4_0(T_k ; Q^{2f}_{k,0})$ has now been reconstructed. ∎

Lemma 8.2. The sufficient condition for a complete triangulation graph, $T_k(S^2)$, *(k≥4),* to be four colorable is the existence of $col^4_0(T_3 ; Q^{2f}_{4,0})$, defined in Theorem 7. ∎

[Proof] If there exists $col^4_0(T_3 ; Q^{2f}_{4,0})$, defined in Theorem 7, by applying $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0}) = f^1 (col^4_0(T_{k-i-1} ; Q^{2f}_{k-i,0}))$, for i = k-4, k-5, …, 1, $col^4_0(T_{k-i} ; Q^{2f}_{k-i+1,0})$ can be reconstructed even if T_{k-i}, $Q^{2f}_{k-i+1,0}$, and $col^4_0(T_{k-i-1} ; Q^{2f}_{k-i,0})$ are unknown. This means that The sufficient condition for for a complete triangulation graph, $T_k(S^2)$, *(k≥4),* to be four colorable is the existence of $col^4_0(T_3 ; Q^{2f}_{4,0})$. ∎

Theorem 9. The necessary and sufficient condition for a complete triangulation graph, $T_k(S^2)$, *(k≥4),* to be four colorable is the existence of $col^4_0(T_3 ; Q^{2f}_{4,0})$, defined in Theorem 7. ∎

[Proof] Evident from Lemma 7.1 and Lemma 8.2. ∎

Lemma 9.1. The necessary and sufficient condition (defined in Theorem 9) for a complete triangulation graph, $T_k(S^2)$, *(k≥4),* to be four colorable, is satisfied. ∎

[Proof] In theorem 9, $col^4_0(T_3 ; Q^{2f}_{4,0})$ really exist as one of the 4x3x2 = 24 colorations with 4 colours. Thus we have reached Lemma 9.1. ∎

Theorem 10. (The Four-Colour Theorem): Every complete triangulation graph, $T_k(S^2)$ (k≥3) is four-colourable. ∎

[Proof] Evident from Lemma 9.1. ∎

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