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Study on Discrete Adiabatic Quantum Computation in 3-SAT Problems

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Abstract

This paper proposes a new variation method for the phase functions of the adiabatic quantum algorithm, quadric variation method. Experiments are carried out solving 3-SAT problem with discrete adiabatic quantum algorithm to compare the proposed two formulation of quadric variation method performance with the previously proposed methods linear and cubic.

Keywords:
Quantum Computation; Adiabatic Theory; Hamiltonian; k-SAT Problem.

I. Introduction

The field of quantum computation have gone through an amazing development in past decade, it has rise as one of the hot area of research after existence of many quantum algorithms showing that the quantum algorithms greatly enhance the efficiency of solving problems believed to be intractable on classical computers. as Peter Shor's polynomial time quantum algorithm for factorizing integers [1], Grover algorithm for unstructured search with quadric speed up over any classical algorithm [2] and Hogg algorithm [3].

The Model of adiabatic quantum computation is a new paradigm for designing quantum algorithms proposed by Farhi et al.[4]. The Adiabatic model is based on quantum adiabatic theorem, where the quantum computer evolves the quantum system slowly to switch gradually from an initial Hamiltonian with ground state easy to construct, to a final Hamiltonian whose ground state encodes the solution of the problem being solved.

In recently published articles, the Adiabatic quantum algorithm, was shown to give polynomial average cost growth for some NP combinatorial search problems as Satisfiability problems [5], and set partitioning problem [6].

This paper compares the adiabatic algorithm performance in solving 3-SAT problems using three different variation methods Linear,cubic, and the proposed method, as result the corresponding search costs and probability of finding the solution are shown.

II. k-SAT Problems

The k-satisfiability problem (k-SAT) is a combinatorial search problem, whose instance is a Boolean expression written using AND, OR, NOT, n variables, and m clauses. A clause is a logical OR of k variables, each of which may be negated. Given an expression, the solution is an assignment ,i.e., a value of TRUE or FALSE values for each variable that will make the entire expression true,i.e., satisfying all the clauses [7]. An example 2-SAT instance with 3 variables and 2 clauses is $(v_1\text{OR}(NOTv_2))$ AND (v_2 OR v_3), which has 4 solutions, for example, v_1 = v_2 = $false$ and v_3 = true. For a given instance,the cost $c(s)$ of an assignment s is the number of clauses it does not satisfy. For $k \geq 3$, k-SAT is NP-complete, i.e., among the most difficult NP problems in the worst case [8].

III. The Quantum Adiabatic Theorem

The adiabatic theorem states that, if the Hamiltonian of any quantum system $H(t)$ varies slowly enough, the state of the system will stay close to the instantaneous ground state of the Hamiltonian at each time t [3]. Assume we can build a Hamiltonian $H^{(c)}$ with ground state encodes the solution of the problem instance to be solved, and prepare the system in the known ground state of another Hamiltonian $H^{(0)}$. Then the adiabatic algorithm can continuously evolve the state of the quantum computer using

$$H(f) = (1 \quad f)H^{(0)} + fH^{(c)} \tag{1}$$

Table I
THE PARAMETERS USED IN DISCRETE AND CONTINUOUS VARIATION METHODS.

Parameters phase functions	Variation methods			
	Linear	Quadric 1	Quadric 2	Cubic
Δ	$1/\sqrt{j}$	const(=1)	const(=1)	const(=1)
Phase shift function	$\rho(f) = f$	$\rho(f) = f^2$	$\rho(f) = 2f - f^2$	$\rho(f) = 1.921f - 2.665f^2 + 1.782f^3$
Phases Mix function	$\tau(f) = 1 - f$	$\tau(f) = 1 - \rho(f)$	$\tau(f) = 1 - \rho(f)$	$\tau(f) = 1 - \rho(f)$

with f ranging from 0 to 1 [3]. Under suitable conditions, i.e., with a nonzero gap between relevant eigenvalues of $H(f)$, and with sufficiently slow changes in f, the adiabatic theorem guarantees that the evolution maps the ground state of $H^{(0)}$ into the ground state of $H^{(c)}$, so a subsequent measurement gives a solution.

IV. THE DISCRETE ADIABATIC ALGORITHM

In this paper, we use the algorithmically equivalent discrete formulation of the adiabatic algorithm acting on the amplitude state vector initially in the ground state of the Hamiltonian $H^{(0)}$, which can be represented as $|\psi_s^{(0)}\rangle = \frac{1}{\sqrt{N}}[1, 1, ..., 1]^T$. Consider a discreet Hamiltonian $H(f)$ of the general form

$$H(f) = \ (f)H^{(0)} + \ (f)H^{(c)} \qquad (2)$$

where (f) and (f) are phase mixing and phase shift function, respectively. both of them are arbitrary functions of f where ($0 \leq f \leq 1$), see Table.1, subject to the boundary conditions

$$(0) = 1, \ \ (0) = 0 \qquad (3)$$

$$(1) = 0, \ \ (1) = 1 \qquad (4)$$

Although, the two functions (f), and (f) are not necessary to be monotonic (i.e. obey the constraint $(f) + \ (f) = 1$), we consider only the monotonic functions [**?**].

In matrix form [3], the Hamiltonian $H^{(c)}$ is a diagonal matrix

$$H_{r,s}^{(c)} = c(s)\delta_{r,s}, \text{where} \quad \delta_{r,s} = \begin{matrix} 1 & \text{if } r{=}s \\ 0 & \text{otherwise} \end{matrix} \qquad (5)$$

This Hamiltonian introduces a phase shift factor in the amplitude of assignment s depending on its associated cost $c(s)$, where the higher cost results in more phase shift. The Hamiltonian $H^{(0)}$ can be implemented with elementary quantum gates by use of the Walsh-Hadamard transform with elements $W_{r.s} = 2 \ ^{n/2}(\ 1)^{r.s}$ [3], where $H^{(0)} = WDW$ and D is a diagonal matrix with the value for state r given by the sum of the bits, i.e, the element $D_{r,r}$ is just a count for the number of bits equal to 1 in state r.

A single trial of the algorithm consists of j steps, parameter Δ and can be described as

1) Initialize the amplitude state vector to the ground state of $H^{(0)}$ giving equal values for all states as $|\psi_s^{(0)}\rangle = \frac{1}{\sqrt{N}}[1, 1, ..., 1]^T$
2) For Steps $h = 1$ through j repeat the matrix multiplication :
$$|\psi^{(h)}\rangle = U_h(f)|\psi^{(h \ \ 1)}\rangle \qquad (6)$$
where $|\psi^{(h \ \ 1)}\rangle$ is the amplitude state vector at step $h \quad 1$, and $U_h(f)$ is unitary evolution operator for h th step which can be represented as
$$U_h(f) = e^{\ i\tau(f)H^{(0)}\Delta/2} \ e^{\ i \ (f)H^{(c)}\Delta} \ e^{\ i\tau(f)H^{(0)}\Delta/2} \qquad (7)$$
3) Measure the final system After the j steps take place, the probability to find a solution is given by $P_{soln} = \sum_s ||\psi^{(j)}||^2$ with the sum over all solutions s.

As a choice for the evolution, f is chosen to vary linearly from 0 to 1. Specifically, we take $f = h/(j+1)$ for step h, where h is ranging from 1 to j.

V. EXPERIMENTS AND CONSIDERATIONS

A. Variation Methods and parameter Δ

Recently several variation methods for phase shift function (f), and phase mixing function (f) were presented as an attempts to decrease the overall search cost [3]. In this paper we present monotonic (means $\ + \ =1$) version of quadric variation method in two formula of quadric polynomial in f. Table .1. summaries this methods , linear , cubic and the two quadric formula we presents, and also it shows the values of the parameter Δ as used with each method.

Figures 1, 2, 3, amd 4 shows the phase fuctions for each variation methoud $vs.$ the f value from 0 to 1, foucasing in the figures we can note that the quadric variation methoud shown in fig. 3, has smalle diversity area than the others whcih gives a sign for expecting better results. A good performance of the discrete adiabatic algorithm requires an appropriate choice of parameter Δ. The experiments have shown that the performance of the algorithm remains good for moderate value of j provided that Δ is below some threshold value. The experiments for solving 3-SAT problems with $n \leq 20$ has shown this threshold to be somewhat near 1.

B. Search behavior

In this section we compare the search behavior of the discrete adiabatic algorithm using the variation methods as summarized in Table.1. First is the linear variation method

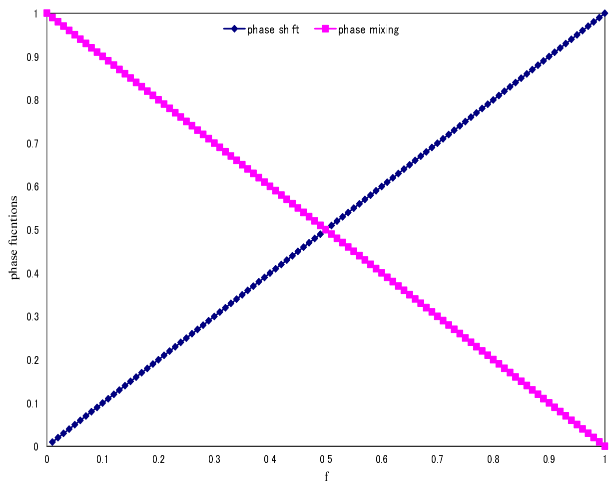

Figure 1. Phase shift and phase mix functions Vs f for the linear variation method.

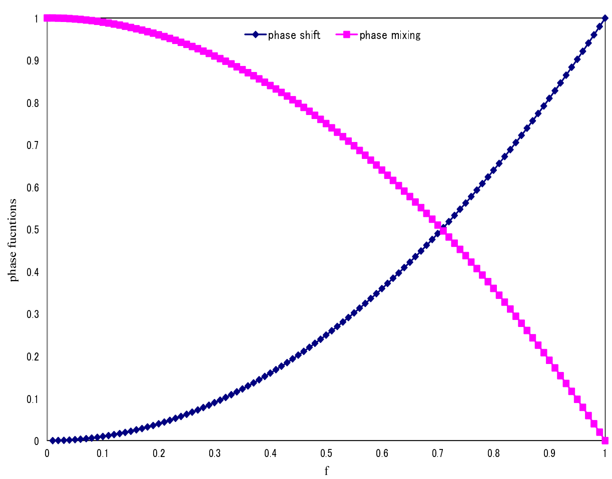

Figure 2. Phase shift and phase mix functions Vs f for the quadric variation method with $\rho = f^2$ formula.

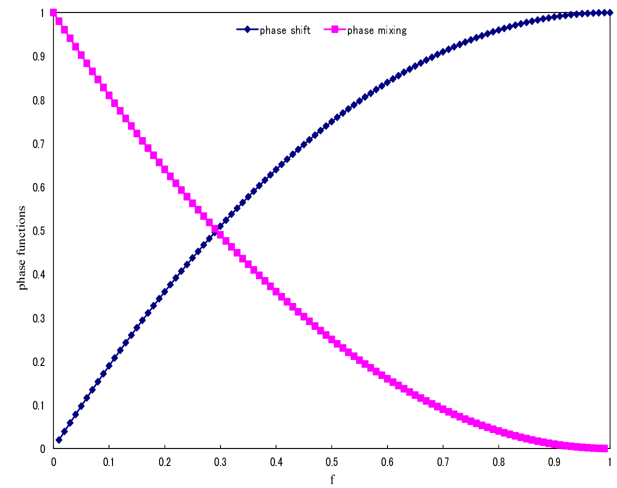

Figure 3. Phase shift and phase mix functions Vs f for the quadric variation method with $\rho = 2f - f^2$ formula.

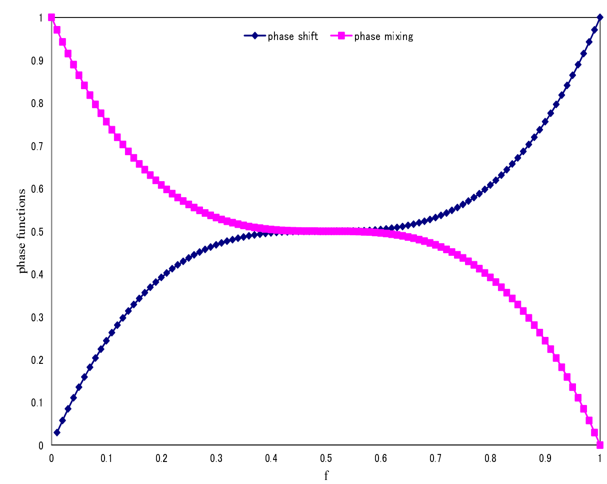

Figure 4. Phase shift and phase mix functions Vs f for the Cubic variation method.

corresponds to the continuous adiabatic algorithm, this method uses $\Delta = 1/\sqrt{j}$, phase shift (f) and phase mixing (f) functions varies monotonically as linear function of f, and number of steps j grows as cubic number of bits n^3. Second method is the cubic variation [**?**] with (f) and (f) varies monotonically as cubic polynomial in f, constant Δ, and uses j grows as square number of bits n^2, and finally the quadric variation method with two formula $(f) = f^2$ and $(f) = 2f \quad f^2$.

Figures 5, and 6 compare the search behavior of the algorithm in solving 3-SAT problem with $n = 8, and 16$, respectively. The figures show that the quadric variation method with number of steps j only as $2 * n$ can achieve P_{soln} near 0.5 , which shows faster search behavior when it is compared with the algorithm behavior using cubic variation with number of steps at least n^2 and the linear variation with $j = n^3$.

C. Search cost

The search cost is defined to be the expected number of steps required to find a solution $C = j/P_{soln}$. Fig. 7 compares the average search cost C for the Linear, cubic, and the two formulation of quadric variation methods for the adiabatic algorithm. it shows that using quadric variation with just enough number of steps $j = 2 * n$ to achieve moderate P_{soln} as shown in Figures. 5, and 6, reduces the search cost below the other methods. However the algorithm with linear variation could achieve P_{soln} near 1 in most of the trials as shown, the number of steps required j grows as n^3 giving a large search costs, far higher than those of other

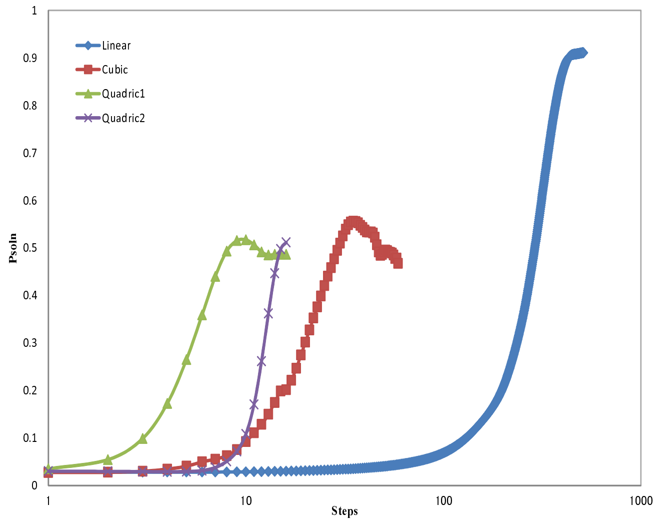

Figure 5. P_{soln} vs number of steps h for the 3 methods averaged over 10 random instances of 3-SAT problems with $n = 8$

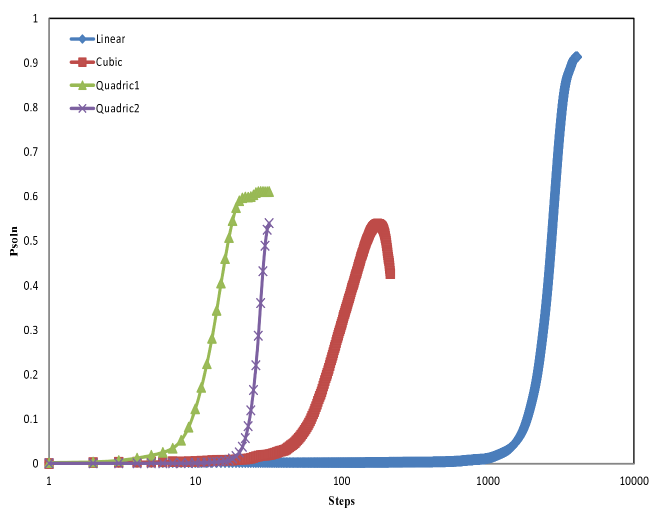

Figure 6. P_{soln} vs number of steps h for the 3 methods averaged over 10 random instances of 3-SAT problems with $n = 16$

method. For comparison the figure shows that the median search cost at $n = 16$ for the quadric1, quadric2, cubic, and linear methods to be 58.07, 56.22, 424.29 and 8749, respectively. which shows improve in the resulting cost reduction using quadric variation method, due to reduced number of steps .

VI. Conclusions

The quantum adiabatic algorithm is a remarkable discovery because it offers new insights into the usefulness of quantum resources for computational tasks. In this paper, we have presented an experimental study in solving 3-

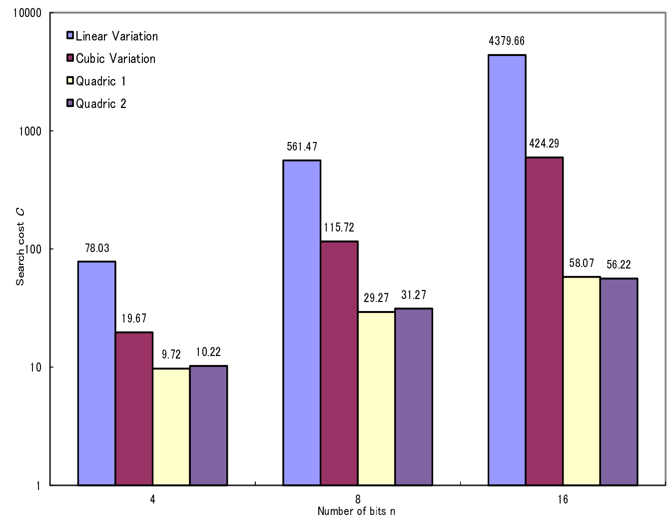

Figure 7. Log plot of the average search cost vs the number of variables n using the same instances in Figures 5 and 6.

SAT problems with the discrete the adiabatic algorithm, using a new monotonic variation method for the phase shift and phase mixing function , quadric variation with two different formula. The experiments have revealed that using the quadric variation method improves on other variation methods, in resulting Search cost and search behavior.

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