Analysis of Navier-Stokes Equation from the Viewpoint of Advection Diffusion (II)
--- Approximate Solution ---

Hiroki Sakamoto [*], Tetsuo Hattori, Akiomi Tada, Vanhoa Nguyen
Graduate School of Engineering, Kagawa University
2217-20, Hayashi-cho Takamatsu, Kagawa 761-0396, Japan

Hiromichi Kawano
NTT Advanced Technology Corporation, 19-18, Nakamachi
Musashino, Tokyo 180-0006, Japan

Abstract

We propose an approximate analysis method for the Navier-Stokes Equation (NSE) based on the similarity between NSE and Advection Diffusion Equation (ADE). In the preceding paper titled "Analysis of Navier-Stokes Equation from the Viewpoint of Advection Diffusion (I)", we have presented the analytical solution of the ADE. Subsequently in this paper, we point out the explicit similarity between NSE and ADE by illustrating the corresponding equations. Then, we show an approximate solution of NSE using the aforementioned analytical solution of ADE.

***Keywords*:** approximate anlysis method, Navier-Stokes Equation (NSE), Advection Diffusion Equation (ADE)

1. Introduction

It is well known that Navier-Stokes Equation (NSE) is the most fundamental one in Fluid Mechanics[1], and also known that the exact analytical solution of NSE is not yet obtained.

Although we have to use numerical computation for solving the NSE under the arbitrary initial and boundary conditions, if possible, it is still desirable to have an analytical approximate solution that is as much as close to the exact one.

In this paper, in order to obtain such an analytical approximate solution, we focus on the similarity between the NSE and Advection Diffusion Equation (ADE). And we apply the exact analytical solution of the ADE over uniform flow field (or velocity field) in three dimensional (3D) boundless region under arbitrary initial condition[2-4], that we have presented in the previous paper of ICAROB2015, to the NSE.

2. Advection Diffusion Equation (ADE)

Let C be a fluid of density (or density of material), and let D_x, D_y, D_z denote the diffusion coefficient in x, y, z axis direction, respectively. Similarly, let u, v, w denote the flow velocity in x, y, z axis direction, respectively. Moreover, let λ and Q be the attenuation coefficient that is spatially uniform and the load generation rate function, respectively.

*2217-20 Hayashi Cho Takamatsu City, Kagawa, 761-0396 Japan,
Kagawa University, Email: hattori@eng.kagawa-u.ac.jp

The partial differential equation of the ADE is shown as Eq.(1).

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2} - u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial y} - w\frac{\partial C}{\partial z} - \lambda C + Q \quad (1)$$

Here we introduce the Dirac's δ function as in the following (2).

$$\begin{cases} (x=0)\ \delta(x)=\infty,\ (x\neq 0)\ \delta(x)=0, \\ \int_{-\infty}^{\infty}\delta(x)dx=1 \end{cases} \quad (2)$$

The initial condition and the load generation rate function are shown in Eq.(3) and Eq.(4), respectively.

$$C(x,y,z,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_0(\xi,\eta,\zeta)\,\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)d\xi d\eta d\zeta \quad (3)$$

$$Q(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi,\eta,\zeta,t)\,\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)d\xi d\eta d\zeta \quad (4)$$

Then, we can obtain the exact solution as follows.

$$C(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_0(\xi,\eta,\zeta)G(x,y,z,t,\xi,\eta,\zeta,0) + \int_0^t Q(\xi,\eta,\zeta,\tau)G(x,y,z,t,\xi,\eta,\zeta,\tau)d\tau\}d\xi d\eta d\zeta \quad (5)$$

where G is a Green function and Gaussian-like exponential one as shown in Eq.(6).

$$G(x,y,z,t,\xi,\eta,\zeta,\tau) = \frac{1}{8\sqrt{\pi^3\int_\tau^t D_x ds\int_\tau^t D_y ds\int_\tau^t D_z ds}} \exp\left\{-\frac{\left(x-\xi-\int_\tau^t u ds\right)^2}{4\int_\tau^t D_x ds} - \frac{\left(y-\eta-\int_\tau^t v ds\right)^2}{4\int_\tau^t D_y ds} - \frac{\left(z-\zeta-\int_\tau^t w ds\right)^2}{4\int_\tau^t D_z ds} - \int_\tau^t \lambda ds\right\} \quad (6)$$

3. Navier-Stokes Equation (NSE)

We consider that the NSE shows a Law of conservation of momentum with respect to $\rho v_i\ (i = x, y, z)$ and that NSE is a kind of Advection Diffusion Equation (ADE) regarding momentum (Eq.(5))

$$\frac{\partial(\rho v_i)}{\partial t} + div\left(-\mu\nabla v_i + \rho v_i \boldsymbol{v}\right) = -\frac{\partial p}{\partial x_i} + f_i \quad (7)$$

where ρ , v , μ , p , f are density, velocity, coefficient of viscosity, pressure, and external force, respectively.

3.1. *Similarity between ADE and NSE*

In Eq.(1), let $D_x = D_y = D_z = D$. Using the relational equation in Eq(8), Eq.(1) can be represented as Eq.(9).

$$(\boldsymbol{v}\cdot\nabla) = (v_x, v_y, v_z)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial y}\right) = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial y} \quad \ldots(8)$$

$$\frac{\partial C}{\partial t} = D\nabla^2 C - (\boldsymbol{v}\cdot\nabla)C - \lambda C + Q$$

$$\ldots(9)$$

From NSE Eq.(7), putting $\mu = k\rho$ where k is coefficient of kinematic viscosity, we have

$$\frac{\partial \boldsymbol{v}}{\partial t} = k\nabla^2 \boldsymbol{v} - (\boldsymbol{v} \bullet \nabla)\boldsymbol{v} - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \qquad (10)$$

At $t = t_0$, we have the following relation.

$$\frac{\partial \boldsymbol{v}(t_0)}{\partial t} = k\nabla^2 \boldsymbol{v}(t_0) - \{\boldsymbol{v}(t_0) \bullet \nabla\}\boldsymbol{v}(t_0) - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \quad (11)$$

Here, we consider that $\boldsymbol{v}(t_0) = \boldsymbol{v}(t_0 - \Delta t)$ where Δt is infinitesimal time parameter. So we have

$$\frac{\partial \boldsymbol{v}(t_0)}{\partial t} = k\nabla^2 \boldsymbol{v}(t_0) - \{\boldsymbol{v}(t_0 - \Delta t) \bullet \nabla\}\boldsymbol{v}(t_0) - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \quad \ldots(12)$$

Then putting $\bar{v} = \boldsymbol{v}(t_0 - \Delta t)$, we have another form of the Eq.(9) and Eq.(10) as follows.

$$\frac{\partial C}{\partial t} = D\nabla^2 C - (\bar{\boldsymbol{v}} \bullet \nabla)C - \lambda C + Q \quad \ldots(13)$$

$$\frac{\partial \boldsymbol{v}}{\partial t} = k\nabla^2 \boldsymbol{v} - (\bar{\boldsymbol{v}} \bullet \nabla)\boldsymbol{v} - \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho} \quad \ldots(14)$$

Since the term $-\lambda C$ in Eq.(9) is attenuation one, letting $\lambda = 0$, we have the following correspondent relation.

$$C \leftrightarrow \boldsymbol{v}, \quad k \leftrightarrow D, \quad Q \leftrightarrow \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho}$$

If we regard the $\bar{v}$ as a kind of constant comparing with $\boldsymbol{v}$, we can obtain the solution of Eq.(14) in the same way as the solution of ADE, using the aforementioned correspondent relation.

3.2. *Application of the Solution of ADE to NSE*

Since the diffusion coefficient D appeared in the Green function in Eq.(4), is invariant with respect to time, we can compute the integration term including the D. Then for the Green function, we have

$$G(x,y,z,t,\xi,\eta,\zeta,\tau) = \frac{1}{8\sqrt{\pi^3[D(t-\tau)]^3}} \exp\left\{ -\frac{\left(x-\xi-\int_\tau^t u ds\right)^2}{4D(t-\tau)} - \frac{\left(y-\eta-\int_\tau^t v ds\right)^2}{4D(t-\tau)} - \frac{\left(z-\zeta-\int_\tau^t w ds\right)^2}{4D(t-\tau)} \right\} \quad \ldots(15)$$

Let $\boldsymbol{r}_p = (x,y,z)$, $\boldsymbol{r} = (\xi,\eta,\zeta)$, that means the position at the present time point and the general position at the past time point, respectively. Since we can regard NSE as a kind of ADE, we can obtain the Green function of the analytical approximate solution of NSE by making an analogy from the solution of ADE, as follows..

$$G(x,y,z,t,\xi,\eta,\zeta,\tau) = \frac{1}{8\sqrt{\pi^3[D(t-\tau)]^3}} \exp\left\{ -\left\| \boldsymbol{r_p} - \left(\boldsymbol{r} + \int_\tau^t \bar{\boldsymbol{v}} ds \right) \right\|^2 \Big/ 2\sigma^2 \right\}$$

where $\bar{\boldsymbol{v}} = (u,v,w)$, $\sigma = \sqrt{2D(t-\tau)}$,

$$K = \frac{1}{8\sqrt{\pi^3[D(t-\tau)]^3}} = \left(\sigma\sqrt{\pi}\right)^{-3} \quad (16)$$

Then we have the approximate solution of NSE using the aforementioned Green function (Eq.(16)) as follows.

$$\boldsymbol{v}(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \boldsymbol{v}_0(\xi,\eta,\zeta)G(x,y,z,t,\xi,\eta,\zeta,0)d\xi d\eta d\zeta + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_0^t \boldsymbol{Q}(\xi,\eta,\zeta,\tau)G(x,y,z,t,\xi,\eta,\zeta,\tau)d\tau\right\} d\xi d\eta d\zeta \quad \text{....(17)}$$

where

$$\boldsymbol{Q} = \frac{gradp}{\rho} + \frac{\boldsymbol{f}}{\rho}$$

This means the velocity (or momentum) at the point $\boldsymbol{r}_p = (x,y,z)$ is decided by taking the diffusion of the momentum from the surrounding points that can be computed by the convolution of the Green function (Eq.(16)).

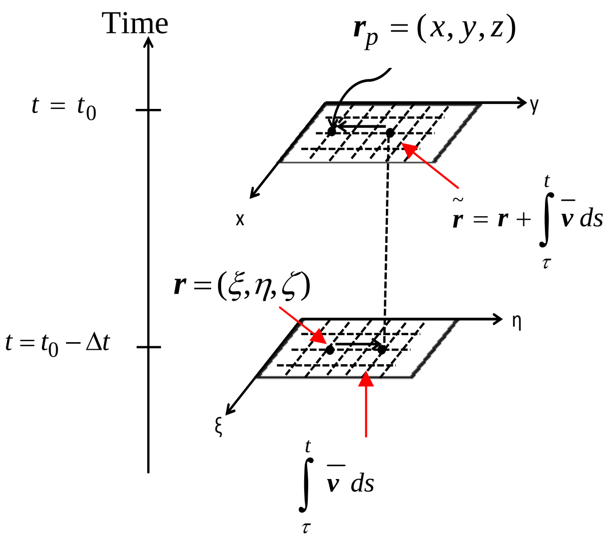

Fig.1. The situation that the present point $\boldsymbol{r}_p$ receives the momentum (or velocity) diffusion from the surrounding points $\tilde{\boldsymbol{r}}$ where a point $\boldsymbol{r}$ at the past time is shifted by the velocity (or momentum).

4. Conclusion

In this paper, focusing on the similarity between the Navier-Stokes Equation (NSE) and Advection Diffusion Equation (ADE), we have described the similarity and pointed out the correspondence relation. Moreover, based on the viewpoint that NSE is a kind of advection (or convection) diffusion equation of momentum, we have derived a solution of NSE by applying the exact analytical solution of the ADE to the NSE.

Although this solution can be regarded as an analytical approximate solution of the NSE, we consider that it infinitely approaches to the exact analytical solution if Δt approaches to 0.

References

1. T. Tanahashi, *Introduction to Computational Fluid Dynamics --Advection Diffusion Equation--*, (in Japanese), (Corona Publishing Co., Ltd., 1996).
2. A.Tada,. An Iteration Method Applying Analytical Solution of the Diffusion Equation, (in Japanese), *Japan Society of Civil Engineering*, vol. 2-26, No. 485, (1994) pp.1-10
3. T.Hattori and A.Tada: A Novel Computational Method for Convection Diffusion Equation under Arbitrary Initial and Boundary Condition, *15th IMACS World Congress on Scientific Computation, Modelling and Applied Mathematics, Edited by A.Sydow*, (Wissenschaft & Technik Verlag, vol.3, Berlin, 1997), pp.645-650.
4. T.Hattori, K.Shinagawa, and A.Tada, Averaged Green Function Method for Diffusion Problem, *Proc. of 1998 International Symposium on Nolinear Theory and Its Applications* (NOLTA'98) 1998, pp.979-982.