Analysis of Navier-Stokes Equation from the Viewpoint of Advection Diffusion (I)
-- Analytical Solution of Diffusion Equation --

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Abstract

We propose an approximate analysis method for Navier-Stokes Equation (NSE) based on the similarity between NSE and Advection Diffusion Equation (ADE). In this paper, we present an analytical solution and a Green function (integral kernel) which are obtained from the diffusion equation over uniform flow field (or velocity field) in three dimensional (3D) boundless region under arbitrary initial condition. The solution shows that the diffusion process is a Markov one and that the Green function becomes a Gaussian-like exponential function.

***Keywords*:** approximate anlysis method, Navier-Stokes Equation (NSE), Advection Diffusion Equation (ADE)

1. Introduction

Navier-Stokes Equation (NSE) is well known as fundamental one in Fluid Mechanics[1]. Because the exact analytical solution of NSE is not yet obtained, we have to use numerical computation for the solution under the arbitrary initial and boundary conditions.

For the numerical computation method, Difference Method, Finite Element Method and Boundary Element Method are well known. However, as for the Difference Method, it tends to be difficult to deal with complicated boundary conditions. In addition, the method has to satisfy the Courant condition to obtain the stable computation solution.

On the other hand, the Finite Element Method takes much time to solve simultaneous equations appeared in the method, and the Boundary Element Method has a problem in the computation precision for the analysis of viscous flow of high Reynolds number.

Even if we obtain the result by using such numerical computations, those methods take much time and the results are involved by not a little computation error.

So, in order to reach the more accurate solution, it would be desirable to have an analytical approximate solution that is as much as close to the exact one.

In order to obtain such an analytical approximate solution, we focus on the similarity between the NSE and Advection Diffusion Equation (ADE). And in this paper, we derive the exact analytical solution of the ADE over uniform flow field (or velocity field) in three dimensional (3D) boundless region under arbitrary initial condition[2-4].

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2. Advection Diffusion Equation

Let C be a fluid of density (or density of material), and let D_x, D_y, D_z denote the diffusion coefficient in x, y, z axis direction, respectively. Similarly, let u, v, w denote the flow velocity in x, y, z axis direction, respectively. Moreover, let λ and Q be the attenuation coefficient that is spatially uniform and the load generation rate function, respectively.

The partial differential equation of the ADE is shown as Eq.(1).

$$\frac{\partial C}{\partial t} = D_x \frac{\partial^2 C}{\partial x^2} + D_y \frac{\partial^2 C}{\partial y^2} + D_z \frac{\partial^2 C}{\partial z^2} - u\frac{\partial C}{\partial x} - v\frac{\partial C}{\partial y} - w\frac{\partial C}{\partial z} - \lambda C + Q \qquad (1)$$

Here we introduce the Dirac's δ function as in the following (2).

$$\begin{cases} (x=0)\ \delta(x)=\infty,\ (x\neq 0)\ \delta(x)=0, \\ \int_{-\infty}^{\infty}\delta(x)dx=1 \end{cases} \qquad (2)$$

The initial condition and the load generation rate function are shown in Eq.(3) and Eq.(4), respectively.

$$C(x,y,z,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_0(\xi,\eta,\zeta)\,\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)\,d\xi d\eta d\zeta \qquad \cdots(3)$$

$$Q(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi,\eta,\zeta,t)\,\delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)\,d\xi d\eta d\zeta \qquad \cdots(4)$$

Next, we think of the Fourier transform for C in 3 dimensional space. Then, let $\tilde{C}$ be the Fourier transformed C, and we have the following Eq.(5) and Eq.(6).

$$\tilde{C} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(x,y,z,t)e^{-j(px+qy+rz)}dxdyxz \qquad \cdots(5)$$

$$C = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{C}(p,q,r,t)e^{+j(px+qy+rz)}dpdqxr \qquad \cdots(6)$$

From the Fourier transform (FT) of the left hand side (LHS) and right hand side (RHS) in Eq.(1), we have the following Eq.(7) and Eq.(8), respectively.

$$\begin{aligned} &\text{FT of LHS in Eq.(1)} \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\partial C(x,y,z,t)}{\partial t}e^{-j(px+qy+rz)}dxdyxz \\ &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(x,y,z,t)e^{-j(px+qy+rz)}dxdyxz \\ &= \frac{\partial \tilde{C}}{\partial t} \end{aligned} \qquad \cdots(7)$$

$$\begin{aligned} &\text{FT of RHS in Eq.(1)} \\ &= -\{p^2 D_x + q^2 D_y + r^2 D_z + j(pu+qv+rw)+\lambda\}\tilde{C} \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(x,y,z,0)\,e^{-j(px+qy+rz)}dxdyxz \\ &= -\{p^2 D_x + q^2 D_y + r^2 D_z + j(pu+qv+rw)+\lambda\}\tilde{C} \\ &+ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi,\eta,\zeta,t)e^{-j(px+qy+rz)}d\xi d\eta d\zeta \end{aligned} \qquad \cdots(8)$$

As a result from the above Eq.(7) and Eq.(8), the Fourier transform of Eq.(1) becomes as follows.

$$\frac{\partial \tilde{C}}{\partial t} = -\alpha(p,q,r,t)\cdot\tilde{C} + \beta(p,q,r,t) \qquad \cdots(9)$$

where

$$\alpha(p,q,r,t)=p^2D_x+q^2D_y+r^2D_z+j(pu+qv+rw)+\lambda$$

$$\beta(p,q,r,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q(\xi,\eta,\zeta,t)e^{-j(p\xi+q\eta+r\zeta)}d\xi d\eta d\zeta$$

Since Eq.(9) is a linear differential equation of first order with respect to time t, using an infinitesimal time parameter $\tau(0\le\tau\le t)$, the solution is represented as shown in Eq.(10).

$$\tilde{C}(p,q,r,t)=\tilde{C}_0\cdot\exp\left(\int_0^t\alpha(p,q,r,t)ds\right)$$
$$+\int_0^t\beta(p,q,r,t)\cdot\exp\left(-\int_\tau^t\alpha(p,q,r,t)ds\right)d\tau\cdots(10)$$

where $\tilde{C}_0$ is represented as the following equation in the same way as β,

$$\tilde{C}_0(p,q,r,0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi,\eta,\zeta)e^{-j(p\xi+q\eta+r\zeta)}d\xi d\eta d\zeta$$

Next, we apply the inverse Fourier transform to Eq.(10). Let the first and second term of the inverse Fourier transform of right hand side be $C_1(x,y,z,t)$ and $C_2(x,y,z,t)$, respectively.

And, let $\overline{X}$ denote $\int_0^t Xds$.

$$C_1=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\tilde{C}_0e^{-\int_0^t\alpha(p,q,r,s)ds}\right)e^{+j(px+qy+rz)}dpdqdr$$
$$=\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi,\eta,\zeta)e^{+j\{p(x-\xi)+q(y-\eta)+r(z-\zeta)\}}$$
$$\left\{e^{-\int_0^t\{p^2D_x+q^2D_y+r^2D_z+j(pu+qv+rw)+\lambda\}}\right\}d\xi d\eta d\zeta dpdqdr$$
$$=\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi,\eta,\zeta)\exp\{-p^2\overline{D_x}+jp(x-\xi-\overline{u})$$
$$-q^2\overline{D_y}+jp(y-\eta-\overline{v})-r^2D_z+jp(z-\zeta-\overline{w})-\overline{\lambda}\}d\xi d\eta d\zeta dpdqdr$$

$$=\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi,\eta,\zeta)\exp\{-\left\{\left\{p\sqrt{\overline{D_x}}\right\}-j\frac{(x-\xi-\overline{u})}{2\sqrt{\overline{D_x}}}\right\}^2$$
$$-\left\{\left\{q\sqrt{\overline{D_y}}\right\}-j\frac{(y-\eta-\overline{v})}{2\sqrt{\overline{D_y}}}\right\}^2-\left\{\left\{r\sqrt{\overline{D_z}}\right\}-j\frac{(z-\zeta-\overline{w})}{2\sqrt{\overline{D_z}}}\right\}^2\}$$
$$*\exp\left\{-\overline{\lambda}-\frac{(x-\xi-\overline{u})^2}{4\overline{D_x}}-\frac{(y-\eta-\overline{v})^2}{4\overline{D_y}}-\frac{(z-\zeta-\overline{w})^2}{4\overline{D_z}}\right\}d\xi d\eta d\zeta dpdqdr$$
$$\cdots(11)$$

First, we perform the integration with respect to p,q,r , then we have the following Eq.(12)

$$C_1=\frac{1}{8\sqrt{\pi^3\overline{D_x}\,\overline{D_y}\,\overline{D_z}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}C_0(\xi,\eta,\zeta)$$
$$\exp\left\{-\overline{\lambda}-\frac{(x-\xi-\overline{u})^2}{4\overline{D_x}}-\frac{(y-\eta-\overline{v})^2}{4\overline{D_y}}-\frac{(z-\zeta-\overline{w})^2}{4\overline{D_z}}\right\}d\xi d\eta d\zeta$$
$$\cdots(12)$$

As for the C_2 , we have the following equations.

$$C_2=\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\int_0^t\beta(p,q,r,t)\cdot\exp\left(-\int_\tau^t\alpha(p,q,r,t)ds\right)d\tau\right)$$
$$e^{j(px+qy+rz)}dpdqdr$$
$$=\frac{1}{8\pi^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_0^t\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Qe^{-j(p\xi+q\eta+r\zeta)}d\xi d\eta d\zeta\right]\cdot e^{-\int_0^t\alpha ds}d\tau\right\}$$
$$e^{j(px+qy+rz)}dpdqdr$$
$$=\frac{1}{8\pi^3}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q\exp\{jp(x-\xi)+jp(y-\eta)+jr(z-\zeta)$$
$$-\int_\tau^t\left[p^2D_x+q^2D_y+r^2D_z+j(pu+qv+rw)+\lambda\right]ds\}$$
$$dpdqdrd\xi d\eta d\zeta d\tau$$
$$=\frac{1}{8\pi^3}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\{-\left(p^2D_x+q^2D_y+r^2D_z\right)$$
$$+jp(x-\xi-\overline{u})+jq(y-\eta-\overline{v})+jr(z-\zeta-\overline{w})-\overline{\lambda}\}$$
$$dpdqdrd\xi d\eta d\zeta d\tau$$

$$= \frac{1}{8\pi^3}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp - \left\{\left\{ p\sqrt{\overline{D_x}} - j\frac{(x-\xi-\overline{u})}{2\sqrt{\overline{D_x}}}\right\}^2 - \left\{ q\sqrt{\overline{D_y}} - j\frac{(y-\eta-\overline{v})}{2\sqrt{\overline{D_y}}}\right\}^2 - \left\{ r\sqrt{\overline{D_z}} - j\frac{(z-\zeta-\overline{w})}{2\sqrt{\overline{D_z}}}\right\}^2\right\}$$

$$* \exp\left\{ -\overline{\lambda} - \frac{(x-\xi-\overline{u})^2}{4\overline{D_x}} - \frac{(y-\eta-\overline{v})^2}{4\overline{D_y}} - \frac{(z-\zeta-\overline{w})^2}{4\overline{D_z}}\right\}$$

$$dpdqdrd\xi d\eta d\zeta d\tau$$

Therefore, we have

$$C_2 = \int_0^t \frac{1}{8\sqrt{\pi^3\overline{D_x}\,\overline{D_y}\,\overline{D_z}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi,\eta,\zeta,\tau)\exp\left\{ - \frac{(x-\xi-\overline{u})^2}{4\overline{D_x}} - \frac{(y-\eta-\overline{v})^2}{4\overline{D_y}} - \frac{(z-\zeta-\overline{w})^2}{4\overline{D_z}} - \overline{\lambda}\right\} d\xi d\eta d\zeta d\tau \quad \cdots(13)$$

By returning the notation " $\overline{}$ " to the original integration representation in Eq.(12) and Eq.(13), we have

$$C_1 = \frac{1}{8\sqrt{\pi^3\int_0^t D_x ds\int_0^t D_y ds\int_0^t D_z ds}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C_0(\xi,\eta,\zeta)\exp\left\{ - \frac{(x-\xi-\int_0^t uds)^2}{4\int_0^t D_x ds} - \frac{(y-\eta-\int_0^t vds)^2}{4\int_0^t D_y ds} - \frac{(z-\zeta-\int_0^t wds)^2}{4\int_0^t D_z ds} - \int_0^t \lambda ds\right\} d\xi d\eta d\zeta \quad (14)$$

$$C_2 = \int_0^t \frac{1}{8\sqrt{\pi^3\int_\tau^t D_x ds\int_\tau^t D_y ds\int_\tau^t D_z ds}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q(\xi,\eta,\zeta,\tau)\exp\left\{ - \frac{(x-\xi-\int_\tau^t uds)^2}{4\int_\tau^t D_x ds} - \frac{(y-\eta-\int_\tau^t vds)^2}{4\int_\tau^t D_y ds} - \frac{(z-\zeta-\int_\tau^t wds)^2}{4\int_\tau^t D_z ds} - \int_\tau^t \lambda ds\right\} d\xi d\eta d\zeta d\tau \quad (15)$$

Because $C = C_1 + C_2$, we have the solution as follows.

$$C(x,y,z,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{C_0(\xi,\eta,\zeta)G(x,y,z,t,\xi,\eta,\zeta,0) + \int_0^t Q(\xi,\eta,\zeta,\tau)G(x,y,z,t,\xi,\eta,\zeta,\tau)d\tau\}d\xi d\eta d\zeta$$

where

$$G(x,y,z,t,\xi,\eta,\zeta,\tau) = \frac{1}{8\sqrt{\pi^3\int_\tau^t D_x ds\int_\tau^t D_y ds\int_\tau^t D_z ds}}\exp\left\{ - \frac{\left(x-\xi-\int_\tau^t uds\right)^2}{4\int_\tau^t D_x ds} - \frac{\left(y-\eta-\int_\tau^t vds\right)^2}{4\int_\tau^t D_y ds} - \frac{\left(z-\zeta-\int_\tau^t wds\right)^2}{4\int_\tau^t D_z ds} - \int_\tau^t \lambda ds\right\} \quad (16)$$

3. Conclusion

In this paper, we have presented the exact analytical solution of the ADE over uniform flow field (or velocity field) in three dimensional (3D) boundless region under arbitrary initial condition.

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