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Fundamentals of Neurodynamics: Statistical neurodynamics and Neural Field Theory

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Biological Decision is Fast and Robust

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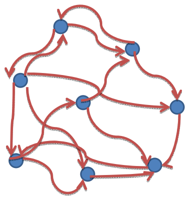


Biological decision

Densely connected
Sparsely connected

Majority
Boolean

$$x_i' = f_i(x_1, x_2, \ldots, x_n)$$

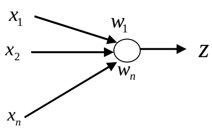
$x_i = 1, -1$ **binary**

Majority Decision vs Boolean Decision

Statistical Neurodynamics

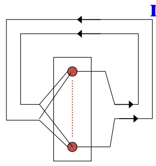
$$z = \mathrm{sgn}(\sum w_i x_i - h_i)$$

$$z = f(x_1, \cdots, x_n)$$



Statistical Neurodynamics

micro-dynamics



$$\mathbf{x} = (x_1, \ldots, x_n) : state$$

$$\mathbf{x}(t+1) = T_w \mathbf{x}(t) = f\{\mathbf{x}(t)\}$$

$$x_i^t = \mathrm{sgn}\left(\sum w_{ij} x_j^{t-1} - h_i\right)$$

macro-dynamics

$$X_{t+1} = F(X_t) \qquad X = \frac{1}{n}\sum x_i$$

Kauffman network: gene regulation

Boolean logic units vs majority decision units
sparsely connected vs densely connected

dilemma: fast (sparse) vs
robust (majority decision)

densely connected majority units (neural networks)

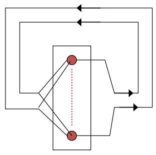
	Boolean	majority
sparse	Kauffman	Kurten
dense	random graph	Amari ○

dilemma:

fast (sparse; dense--chaos)
robust (majority decision)

densely connected majority units

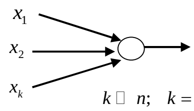
majority decision logic : Boolean logic



$$x_i' = \operatorname{sgn}\left(\sum w_{ij} x_j - h_i\right)$$

$$x_i' = f(x_1, x_2, \ldots x_n)$$

densely connected : sparsely connected



$k \ll n$; $k = 2,3,4$

Statistical neurodynamics

$$X = \frac{1}{n}\sum x_i$$

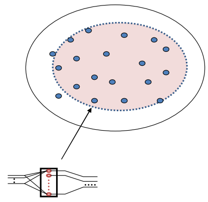
$$X' = F(X) = \operatorname{erf}(\overline{w}X - \overline{h}):$$

$$X(t+1) = F(X(t))$$

mono-stable; bistable

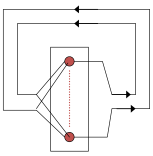
Ensemble of neural networks

The laws
that hold for almost all networks in the ensemble



$$\Omega = \{\omega\} = \{W, h\}$$

Statistical Neurodynamics



micro-dynamics

$$\mathbf{x}(t+1) = T_w \mathbf{x}(t) = \operatorname{sgn}(W\mathbf{x}(t))$$

macro-dynamics $X = \frac{1}{n}\sum x_i$

$$X_{t+1} = F(X_t)$$

$$X_2 = X(\mathbf{x}_2) = X(T_W \mathbf{x}_1)$$

$$X_3 = X(\mathbf{x}_3) = X(T_W T_W \mathbf{x}_1)?$$

Strong proposition of macrodynamics

$$\lim_{n\to\infty}\sup_t E\left[\left|X_t - \tilde{X}_t\right|^2\right] = 0 \qquad X_t = X(T_\omega^t \mathbf{x}_0)$$

$$\tilde{X}_{t+1} = F\left(\tilde{X}_t\right) : \qquad \boldsymbol{x}_0$$

weak proposition of macrodynamics

$$\sup_t \lim_{n\to\infty} E\left[\left|X_t - \tilde{X}_t\right|^2\right] = 0,$$

Fundamental Problems of Statistical Neurodynamics --- unsolved

$$x_i^t = \operatorname{sgn}\left(\sum w_{ij} x_j^{t-1} - h_i\right) \qquad x_i = \pm 1$$

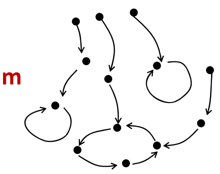
$$\boldsymbol{x}^t = \operatorname{sgn}\left(W\boldsymbol{x}^{t-1} - \boldsymbol{h}\right) = T_\omega \boldsymbol{x}^{t-1}$$

$$X' = F(X) = F(X(T_\omega(\mathbf{x})))$$

$$X_{t+1} = F(X_t)? \quad F(X(T_\omega T_\omega(\mathbf{x})))$$

$$\omega = (W, \boldsymbol{h})$$

State transition diagram



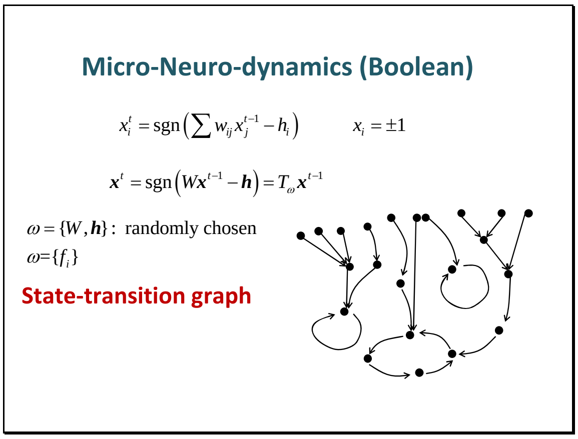
$$X_t = X\left(T_\omega^t \boldsymbol{x}_0\right)$$

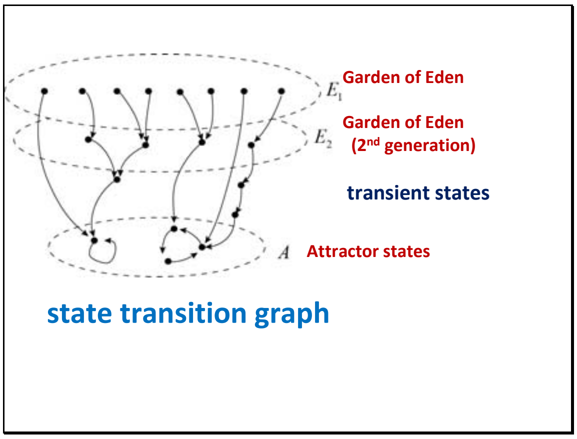
$$X_1 = F\left\{X\left(T_\omega \boldsymbol{x}_0\right)\right\}, \qquad \boldsymbol{x}_1 = T_\omega \boldsymbol{x}_0$$

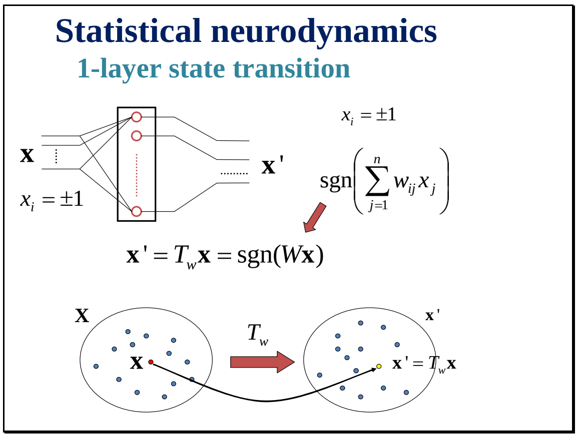
$$X_2 = F\left\{X\left(T_\omega \boldsymbol{x}_1(\omega)\right)\right\} \quad ?$$

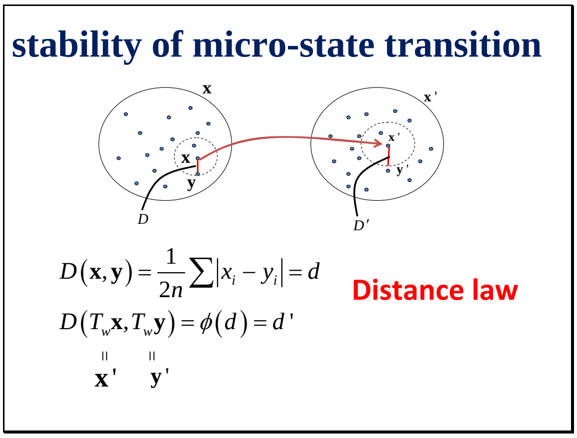
Boltzman H theorem, Kac

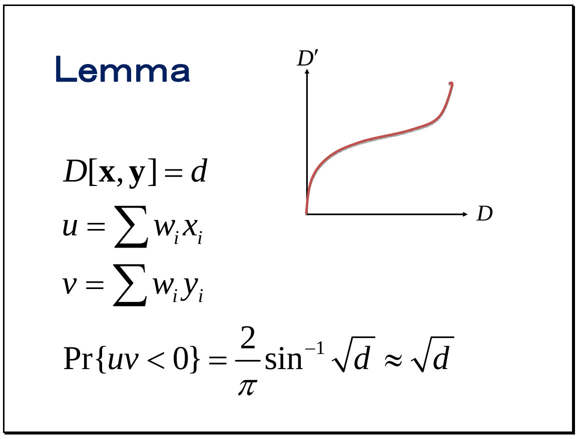
Associative memory

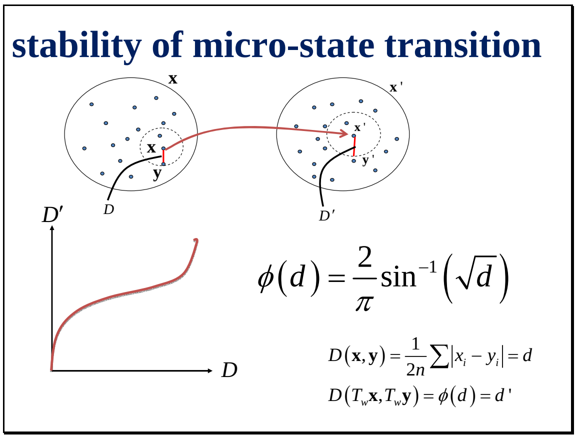

Micro-Neuro-dynamics (Boolean)
$x_i^t = \mathrm{sgn}\left(\sum w_{ij} x_j^{t-1} - h_i\right)$ $x_i = \pm 1$
$\boldsymbol{x}^t = \mathrm{sgn}\left(W\boldsymbol{x}^{t-1} - \boldsymbol{h}\right) = T_\omega \boldsymbol{x}^{t-1}$
$\omega = \{W, \boldsymbol{h}\}$: randomly chosen
$\omega = \{f_i\}$
State-transition graph

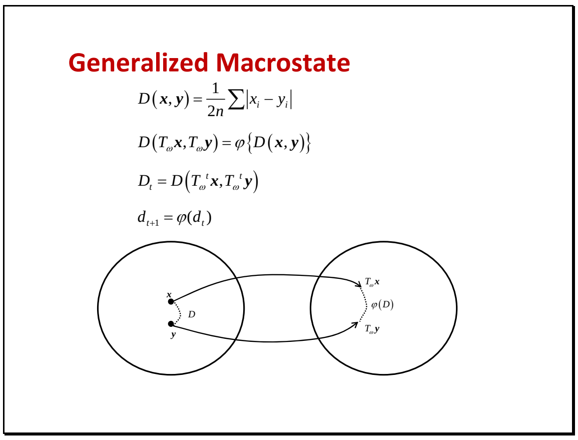

E_1 Garden of Eden
E_2 Garden of Eden
(2nd generation)
transient states
A Attractor states
state transition graph

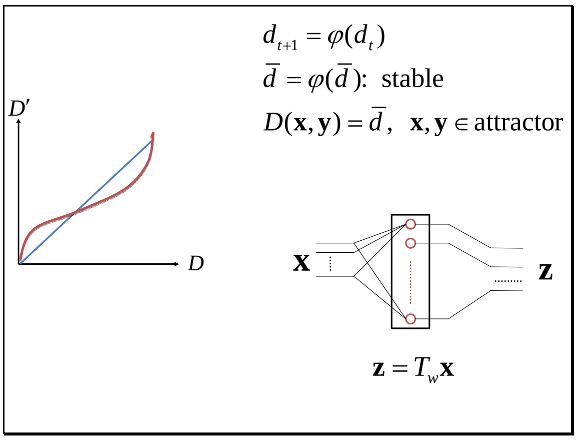

Statistical neurodynamics
1-layer state transition
$\mathbf{x}$
$x_i = \pm 1$
$\mathbf{x}'$
$x_i = \pm 1$
$\mathrm{sgn}\left(\sum_{j=1}^{n} w_{ij} x_j\right)$
$\mathbf{x}' = T_w \mathbf{x} = \mathrm{sgn}(W\mathbf{x})$
$\mathbf{X}$
T_w
$\mathbf{x}'$
$\mathbf{x}$
$\mathbf{x}' = T_w \mathbf{x}$


stability of micro-state transition
$\mathbf{x}$ $\mathbf{y}$ D
$\mathbf{x}'$ $\mathbf{y}'$ D'
$D(\mathbf{x}, \mathbf{y}) = \frac{1}{2n} \sum |x_i - y_i| = d$
$D(T_w \mathbf{x}, T_w \mathbf{y}) = \phi(d) = d'$
$\mathbf{x}'$ $\mathbf{y}'$
Distance law

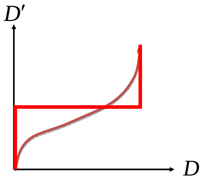

Lemma
D'
D
$D[\mathbf{x}, \mathbf{y}] = d$
$u = \sum w_i x_i$
$v = \sum w_i y_i$
$\Pr\{uv < 0\} = \frac{2}{\pi} \sin^{-1} \sqrt{d} \approx \sqrt{d}$


stability of micro-state transition
$\mathbf{x}$ $\mathbf{y}$ D
$\mathbf{x}'$ $\mathbf{y}'$ D'
D'
D
$\phi(d) = \frac{2}{\pi} \sin^{-1}\left(\sqrt{d}\right)$
$D(\mathbf{x}, \mathbf{y}) = \frac{1}{2n} \sum |x_i - y_i| = d$
$D(T_w \mathbf{x}, T_w \mathbf{y}) = \phi(d) = d'$


Generalized Macrostate
$D(x, y) = \frac{1}{2n} \sum |x_i - y_i|$
$D(T_\omega x, T_\omega y) = \varphi\{D(x, y)\}$
$D_t = D(T_\omega^t x, T_\omega^t y)$
$d_{t+1} = \varphi(d_t)$
x y D
$T_\omega x$ $\varphi(D)$ $T_\omega y$


$d_{t+1} = \varphi(d_t)$
$\bar{d} = \varphi(\bar{d})$: stable
$D(\mathbf{x}, \mathbf{y}) = \bar{d}$, $\mathbf{x}, \mathbf{y} \in$ attractor
D'
D
$\mathbf{x}$
$\mathbf{z}$
$\mathbf{z} = T_w \mathbf{x}$

Distance Laws



$$\phi_M(d) = \frac{2}{\pi}\sin^{-1}\left(\sqrt{d}\right)$$

$$\phi_B(d) = 1/2, \quad d \neq 0; \quad =0, \quad d = 0$$

$$\phi_{SB,k}(d) = \frac{1}{2}\{1-(1-d)^k\}$$

$$\phi_{SM,k}(d) = \sum f_{k,i} c_i(d)$$

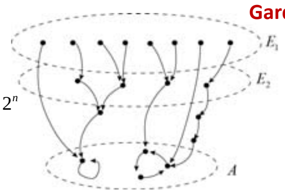
$$T_t = E_1 \cup E_2 \cup \cdots \cup E_r \to \overline{T} = T_r$$

Gardens of Eden

$X = \{\mathbf{x}\}; \quad |X| = 2^n$

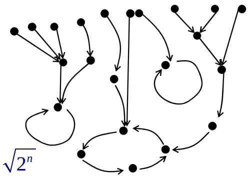
$X = \overline{T} \cup A$

$A = \bigcup C_i$



state transition graph

state –transition graph---Boolean --- random graph



$N = 2^n$ states

number of attractor states : $\sqrt{2^n}$

number of attractor cycles : n

transient length : $\sqrt{2^n}$

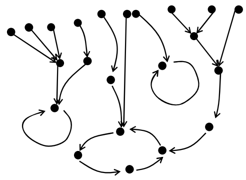
microscopic dynamics

$x_i^t = \mathrm{sgn}\left(\sum w_{ij} x_j^{t-1} - h_i\right) \qquad x_i = \pm 1$

$\mathbf{x}^t = \mathrm{sgn}\left(W\mathbf{x}^{t-1} - \mathbf{h}\right) = T_\omega \mathbf{x}^{t-1}$

$\omega = (W, \mathbf{h})$

state transition diagram



Concentration of wealth (monopoly!) --- speed of convergence

$\mathbf{s}_0 = (s_{0\alpha}) = (1,1,1,\ldots,1)$

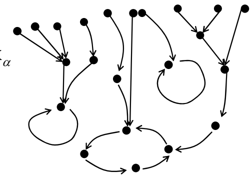
$T_\omega = (T_{\alpha\beta}); \quad T_{\alpha\beta} = 1 \text{ when } \mathbf{x}_\beta \to \mathbf{x}_\alpha$

$\mathbf{s}_1(\omega) = T_\omega \mathbf{s}_0(\omega)$

$\overline{S}_1 = E[s_{1\alpha} \mid \mathbf{x}_\beta \to \mathbf{x}_\alpha]$

$\mathbf{s}_t(\omega) = T_\omega^{\,t} \mathbf{s}_0(\omega):$

$\overline{S}_t = E[s_{1\alpha} \mid \mathbf{x}_\beta \to \mathbf{x}_{\beta'} \ldots \to \mathbf{x}_\alpha]$



:Expected number of coins after t state transitions

$\overline{S}_1$:Expected number of in-coming branches when x is not in the garden of Eden

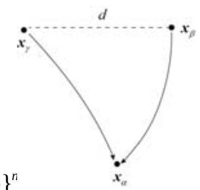
Concentration index

$$\alpha_k = \lim_{n\to\infty} \frac{1}{n} \log \overline{S}_k$$

$$\overline{S}_k \approx 2^{\alpha_k n}$$

$\alpha_k = 0$ for densely connected Boolean net

Calculation of $\overline{S}_1$



$\mathbf{x}_\alpha = T_\omega \mathbf{x}_\beta$

$D(\mathbf{x}_\beta, \mathbf{x}_\gamma) = d$

$\Pr\left\{T_\omega \mathbf{x}_\gamma = \mathbf{x}_\alpha \mid \mathbf{x}_\alpha = T_\omega \mathbf{x}_\beta\right\} = \{1-\varphi(d)\}^n$

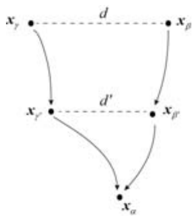
$\overline{S}_1 = \sum {}_n C_{nd} \{1-\varphi(d)\}^n$

$= \int \exp\{n[H(x) + \log\{1-\varphi(x)\}]\}dx$

$= \exp[n[H(x_0) + \log\{1-\varphi(x_0)\}]$

$x_0 = \arg_x \max\{H(x) + \log\{1-\varphi(x)\}$

Calculation of $\overline{S}_2$



$\mathbf{x}_\beta \to \mathbf{x}_{\beta'} \to \mathbf{x}_{\beta''}$

$\mathbf{x}_\gamma \to \mathbf{x}_{\gamma'} \to \mathbf{x}_{\gamma''}$

$D(\mathbf{x}_\gamma, \mathbf{x}_\beta) = d; D(\mathbf{x}_{\gamma'}, \mathbf{x}_{\beta'}) = d'$

$d \to d' \to 0$

$\varphi(d)^{nd'}\{1-\varphi(d)\}^{n(1-d')}: \qquad \{1-\varphi(d')\}^n$

$\overline{s}_2 = \int \exp\{n[H(x) + H(x') + d'\varphi(x)$

$+ (1-x')\log\{1-\varphi(x)\} + \log\{1-\varphi(x')\}]\}dxdx'$

$$\bar{S}_t = \exp(\alpha_t n)$$
$$\alpha_1 = 0.23$$
$$\alpha_2 = 0.30$$
$$\alpha_3 = 0.32$$

The distribution of in-coming branches: Power law $p_k \approx \frac{1}{k^\gamma}$

$p_k = \Pr\{\#\text{incoming branches}=k\}$

#outgoing branches = 1

$$\sum p_k = 1$$
$$\sum k p_k = 1$$
$$\sigma^2 = \sum k^2 p_k - 1 \to \infty: \quad \text{monopoly}$$
$$p_k \approx \frac{1}{k^\gamma};$$

Friend gathering theorem

$$r_k = \text{Prob}\left\{\left|T_\omega^{-1}\mathbf{x}\right| = k \mid x \notin E_1\right\}$$
$$= \text{Prob}\left\{\left|T_\omega^{-1}\mathbf{x}\right| = k \mid \; T_\omega \boldsymbol{y} = \boldsymbol{x}\right\}$$
$$= \frac{\text{Prob}\left\{\left|T_\omega^{-1}\mathbf{x}\right| = k\right\} \text{Prob}\{T_\omega \boldsymbol{y} = \mathbf{x} \mid \left|T_\omega^{-1}\mathbf{x}\right| = k\}}{\text{Prob}\{T_\omega \boldsymbol{y} = \boldsymbol{x}\}}$$
$$\text{Prob}\left\{T_\omega \boldsymbol{y} = \boldsymbol{x} \mid \left|T_\omega^{-1}\mathbf{x}\right| = k\right\} = \frac{k}{N}$$
$$r_k = k p_k = k \text{Prob}\left\{\left|T_\omega^{-1}\mathbf{x}\right| = k\right\}$$

how to calculate σ^2

$$\sigma^2 = E\left[\left|T_\omega^{-1}\mathbf{x}\right| \;\; \left|T_\omega \boldsymbol{y} = \mathbf{x}\right.\right] - 1$$
$$\text{Prob}\left\{T_\omega \boldsymbol{y} = \mathbf{x} \mid \left|T_\omega^{-1}\mathbf{x}\right| = k\right\} = \frac{k}{N}$$
$$p_k = \text{Prob}\left\{\left|T_\omega^{-1}\mathbf{x}\right| = k, T_\omega \boldsymbol{y} = \mathbf{x}\right\} N$$
$$\sigma^2 = \sum_n {}_nC_k \left\{1 - \phi\left(\frac{k}{n}\right)\right\}^n - 1 = \bar{S}_1$$

The Power law

p_k : probability that a node has k incoming branches

$$\sum k p_k = 1$$
$$\sum k^2 p_k = 2^{\alpha_1 n} \to \infty$$
$$p_k \approx \frac{1}{k^\gamma}; \qquad \gamma = \frac{\alpha_1 + 3}{\alpha_1 + 1} = 2.6$$

Random Boolean graph

$$p_k = \frac{1}{ek!}:$$

Poisson dustribution

$$\sum k p_k = 1$$
$$\sum k^2 p_k = 2$$

State transition of generalized majority decision: fast and robust

very small number of attracting states

small-world network
short transient states
chaotic dynamics

general statements concerning majority decision logic

Neural Oscillation

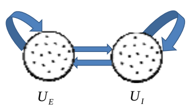
Wiener ・・・ entrainment

$$\frac{dU_E}{dt} = -U_E + w_{EE} f(U_E) - w_{EI} f(U_I) + S_E$$
$$\frac{dU_I}{dt} = -U_I + w_{IE} f(U_E) - w_{II} f(U_I) + S_I$$

f : sigmoidal function

Amari (1971; 1972)

Wilson-Cowan (1972)



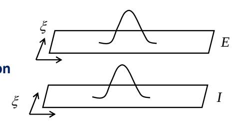
multi-stability
oscillation

Neural Fields : Wilson-Cowan (1972)

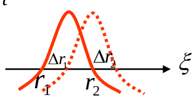
$$\tau_E \frac{\partial U_E(\xi,t)}{\partial t} = -U_E + w_{EE} \circ f(U_E) - w_{EI} \circ f(U_I) + S_E(\xi,t)$$
$$\tau_I \frac{\partial U_I(\xi,t)}{\partial t} = -U_I + w_{IE} \circ f(U_E) - w_{II} \circ f(U_I) + S_I(\xi,t)$$

$\circ$: convolution

Ermentrout-Cowan (1979): hallucination



$$\frac{\partial u(\xi,t)}{\partial t} = -u(\xi,t) + \int w(\xi-\xi')f\left[u(\xi',t)\right]d\xi' + a$$



$$u(r_i,t)=0 \qquad u(r_i+\Delta r_i, t+\Delta t)=0$$

$$\frac{\partial u(r_i,t)}{\partial t} + \frac{\partial u(r_i,t)}{\partial \xi}\frac{\partial r_i}{\partial t} = 0$$

$$\alpha_i \frac{\partial r_i}{\partial t} = -\frac{\partial u(r_i,t)}{\partial t} = \int_{r_1}^{r_2} w(r_i-\xi')d\xi' = \pm W(r_2-r_1)$$

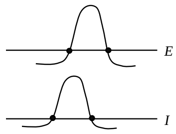
$$W(\xi) = \int_0^{\xi} w(\xi)d\xi$$

Neural Fields ･･･ mathematical theory

Amari (1977) :

dynamics of boundary

$f(u)=1(u)$



- bump solution
- traveling bump
- maximum detector
- working memory

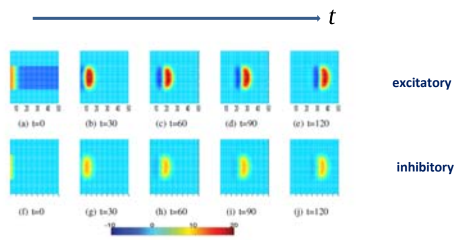
Amari-Kishimoto (1979)

$$\frac{\partial u(\xi,t)}{\partial t} = -u(\xi,t) + \int w\circ f[u(\xi',t)-h]d\xi' - v(\xi,t) + I_u(\xi,t)$$

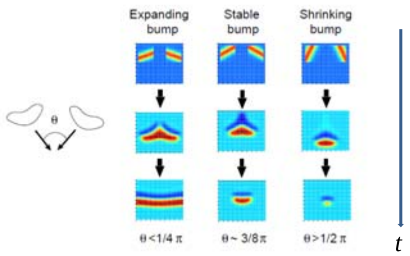
$$\frac{\partial v(\xi,t)}{\partial t} = -\alpha u(\xi,t) - \beta v(\xi,t) + I_v(\xi,t)$$

Pinto-Ermentrout model (2001)

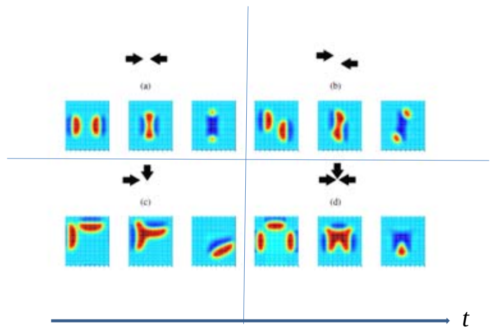
Traveling bump



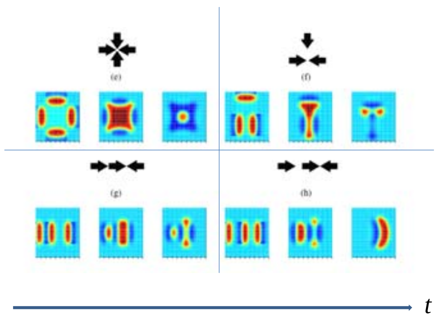
Collision of bumps (1)



Collision of bumps (2)



Collision of bumps (3)



Control of moving bumps