

# Analytical approach to synchrony between populations of neurons with modulatory effects

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**Abstract:** We have analyzed the synchrony of neurons using the effective input theory. First, the dependence of intra-regional synchrony on the strength of synaptic connections is clarified. When the synaptic connections are weaker than thresholds of neurons, spontaneous firings do not exist. Second, we have studied the inter-regional synchrony of two regions connected by modulatory common noises. As a result, under the appropriate modulatory-effects there exist nontrivial sets of synchronized states of regions. Furthermore, at least one of these two regions satisfies the condition of intra-regional synchrony. When both regions do not satisfy the condition of intra-regional synchrony, these two regions should have the same structure of synaptic connections in order to synchronize inter-regionally.

**Keywords:** Populations of neurons, Effective input theory, Synchronized firing, Synchrony between regions, Analytical approach

## 1 INTRODUCTION

Up to now, the input trains to a neuron are mainly described as stochastic processes in such as Ricciardi[1], Stein[2], and others. Because the properties of the stochastic processes with random Langevin forces are well known (Uhlenbeck and Ornstein[3]), we can easily obtain a lot of essential properties of neurons with applying the stochastic theories. However, methods of stochastic theories approximate the neurons to a single neuron framework which corresponds to the mean-field theory or the phenomenological theory. Essentially these approximations do not include synaptic connections directly. On the other hand, Chen and Jasnow[4] formulate the "effective input theory", which includes one or more synaptic connections using "effective inputs". Here, the "effective inputs" denote the mean value of inputs to populations of several neurons, namely "cluster", from outside neurons through the synaptic connections. Thus the effective input theory is useful to clarify the effects of synaptic connections on synchronies of neurons.

Using the effective input theory, we have studied the conditions of the synchrony in two typical cases as follows:

(1) When neurons are firing spontaneously (and periodically) without external inputs from outside the system, there assumes to exist only effective inputs. First, we obtain the frequency of synchronous firing of neurons under effective inputs. Then we require the condition that the effective inputs corresponds to the frequency of synchronous firings. From this condition, namely "self-consistency", we obtain the frequency of the synchronous firing. If the self-consistency is not fulfilled by any frequency, the synchronous firing does not occur. From the above discussion, we have shown that

high frequency is obtained for strong synaptic connections, while low frequency is obtained for weak synaptic connections.

(2) When the neurons additionally receive modulatory effects, we have examined the effects on synchronies of neurons. We have introduced two independent clusters of neurons and obtained the self-consistency using the effective input theory. Here the synaptic connections and external inputs have changed under the modulatory effects. If we assume that modulatory effects are global, two independent clusters receiving the same modulatory effects correlate each other. Then we analyzed the condition of synchrony between the independent areas.

In section 2, we make a brief review of the effective input theory and discuss synchronized firings of neurons in a cluster. Then we obtain the conditions between synaptic connections and thresholds for the spontaneous synchronized firings. In Section 3, we apply the effective input theory for synchronization between different regions with modulatory common noises. In Section 4, we summarize the results of our studies.

## 2 EFFECTIVE INPUT THEORY

In this section, we make a brief review of the effective input theory[4] for discussing a periodic synchronized firing of neurons located in the same cortex region, namely "spontaneous firings", using our formulation.

At first, we approximate the neuron connections as shown in Fig.1. Because the neurons are connected very complexly, we approximate the cluster neurons  $i$  and  $j$  whose membrane potentials are denoted as  $V_i(t)$  and  $V_j(t)$ , respectively, with

the effective input  $I_{\text{eff}}$ . The effective input  $I_{\text{eff}}$  includes large numbers of inputs from outside of the cluster.

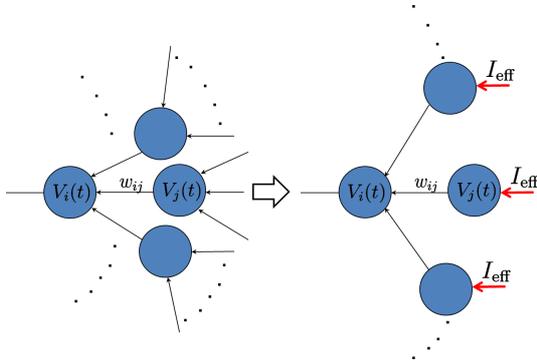


Fig. 1. Approximation of the cluster neurons: The neurons are connected each other as shown in the left-side figure. We approximate the neuron connections to the cluster including the neurons  $i$  and  $j$  and the effective input as shown in the right-side figure.

Then we obtain the effective equations of motion about the membrane potentials  $V_i(t)$  and  $V_j(t)$  as follows:

$$\tau \frac{d}{dt} V_j(t) = -V_j(t) + I_{\text{eff}} \quad (1)$$

and

$$\tau \frac{d}{dt} V_i(t) = -V_i(t) + \sum_{j=1}^c \tau w_{ij} \sum_k \delta(t - t_j^k), \quad (2)$$

where the parameters  $\tau$ ,  $w_{ij}$ ,  $c$ , and  $t_j^k$  denote time-constants, synaptic weights, connection numbers, and the  $k$ -th firing time of neuron  $j$ , respectively.

From Eq.(1), the membrane potential  $V_j(t)$  is obtained as

$$V_j(t) = I_{\text{eff}}(1 - e^{-t/\tau}). \quad (3)$$

Then we obtain the firing time  $t_j^k = kT_j$  using the effective input  $I_{\text{eff}}$  as

$$T_j = -\tau \log \frac{I_{\text{eff}} - \theta}{I_{\text{eff}}} \quad (4)$$

with the threshold potential  $\theta$ . Here, for convenience of calculations, we use a simple condition that the resting potential and the reset potential after firing take the same value of 0.

The time dependence of  $V_i(t)$  is derived from Eq.(2) under the firing of  $j$ -neuron satisfying Eq.(4) as follows:

$$\begin{aligned} V_i(t) &= \frac{1}{\tau} e^{-t/\tau} \int_0^t ds e^{s/\tau} \sum_{j=1}^c \tau w_{ij} \sum_k \delta(s - t_j^k) \\ &= W \frac{1 - e^{-t/\tau}}{1 - e^{-T_j/\tau}}, \end{aligned} \quad (5)$$

where the parameter  $W = \sum_j w_{ij}$  means the total synaptic weight. Then, we obtain the cycle-time  $T_i$  of  $i$ -neuron's firings as

$$T_i = -\tau \log \left[ 1 - \frac{\theta}{W} (1 - e^{-T_j/\tau}) \right]. \quad (6)$$

Now, we consider the self-consistency

$$\begin{aligned} I_{\text{eff}} &= \frac{1}{T} \int_0^T ds \sum_{j=1}^c \tau w_{ij} \sum_{k(<t)} \delta(s - t_j^k) \\ &\simeq \frac{\tau}{T} \sum_{j=1}^c w_{ij} \sum_{k=1}^{T/T_i} \int_0^{T/T_i} ds \delta(s - kT_i) \\ &= \frac{\tau}{T_i} W. \end{aligned} \quad (7)$$

This consistency means a periodic synchronized firing of neurons what we call "spontaneous firing". Then, from Eqs.(4), (6), and (7), we obtain the self-consistent equation as

$$\frac{\tau}{T} (1 - e^{-T/\tau}) = \left( \frac{\theta}{W} \right)^2, \quad (8)$$

where we have redefined  $T = T_i$ . The cycle-time  $T$  of spontaneous firing of the cluster neurons is given as a solution of Eq.(8). The function  $f(T/\tau)$  defined as the left-side of Eq(8), namely  $f(T/\tau) = (\tau/T)(1 - e^{-T/\tau})$ , is plotted in Fig.2. As is shown in Fig.2, the function  $f(T/\tau)$  has the asymptotic value 1 in the case of  $T \rightarrow 0$  (namely the frequency  $\nu = 1/T \rightarrow \infty$ ). Consequently, in the case of  $\theta > W$ , there does not exist the spontaneous firing. On the other hand, in the case of  $\theta < W$ , there exists the spontaneous firing. This result is supported by the following physical phenomena, that is, the firing frequency of neurons are enhanced by effective inputs (from neighbor neurons) exceeding the threshold potential. Meanwhile the spontaneous firing does not occur under the weak effective inputs.

### 3 SYNCHRONY OF DIFFERENT REGIONS

We apply the above effective input theory to analyze the modulatory effects for synchronization between different regions. Membrane potentials of neurons are modulated by diffuse modulatory systems, for example, norepinephrine, 5-hydroxytryptamine, dopamine, and others. These transmitters cause diffusive effects to neurons, and modulate the neuron's behaviors (especially the acceptance of noise effect). Thus we introduce a model which has two neural clusters and the common random input trains from modulators, as shown in Fig.3.

At first, we analyze the membrane potentials  $V_i(t)$  and  $V_j(t)$  in the region R1. The equations of motion of neurons in the region R1 are defined as follows:

$$\tau \frac{d}{dt} V_j(t) = -V_j(t) + I_{\text{eff}} + \eta(t) \quad (9)$$

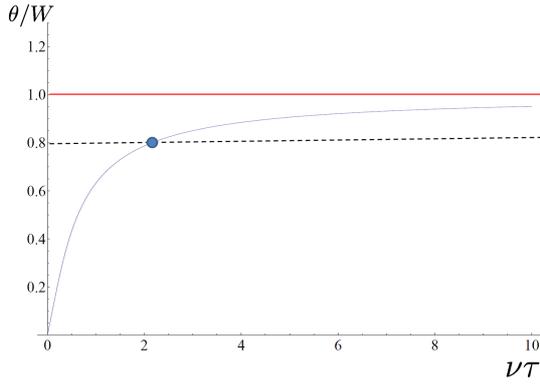


Fig. 2. Condition of spontaneous firings: The vertical axis expresses the rate  $\theta/W$  while the horizontal axis expresses the firing frequency. The condition has the asymptotic value 1 with respect to  $\theta/W$ . Then, in the case of  $\theta > W$ , spontaneous firing never occurs.

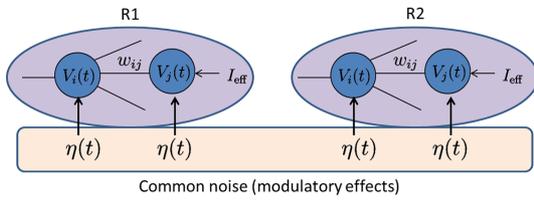


Fig. 3. Modulating effects for populations of neurons: Neurons in the two different regions (R1 and R2) receive the common noise from the diffuse modulatory systems. The regions R1 and R2 synchronize through the common noise.

and

$$\tau \frac{d}{dt} V_i(t) = -V_i(t) + \sum_{j=1}^c \tau w_{ij} \sum_{k(<t)} \delta(t - t_j^k) + \eta(t), \quad (10)$$

using the effective input  $I_{\text{eff}}$  and the common random input  $\eta(t)$ . Here the random input assumes to be a stochastic input train with the average  $\langle \eta(t) \rangle = \lambda$ .

From Eq.(9), the average potential  $\langle V_j(t) \rangle$  is obtained as

$$\langle V_j(t) \rangle = (I_{\text{eff}} + \lambda)(1 - e^{-t/\tau}). \quad (11)$$

Then, the condition  $\langle V_j(T_j) \rangle = \theta$  yields the cycle-time  $T_j$  as

$$T_j = -\tau \log \frac{I_{\text{eff}} + \lambda - \theta}{I_{\text{eff}} + \lambda}. \quad (12)$$

Similarly to the derivation of Eq.(5), we obtain the time dependence of  $V_i(t)$  from Eq.(10) as

$$\begin{aligned} V_i(t) &= \frac{1}{\tau} e^{-t/\tau} \int_0^t ds e^{s/\tau} \left[ \sum_{j=1}^c \tau w_{ij} \sum_{k(<t)} \delta(s - kT_j) + \eta(s) \right] \\ &= W e^{-t/\tau} \frac{e^{T_j/\tau} (e^{t/\tau} - 1)}{e^{T_j/\tau} - 1} + \frac{1}{\tau} e^{-t/\tau} \int_0^t e^{s/\tau} \eta(s) ds. \end{aligned} \quad (13)$$

Then, the random average of Eq.(13) yields

$$\langle V_i(t) \rangle = \sum_{j=1}^c w_{ij} e^{-t/\tau} \frac{e^{T_j/\tau} (e^{t/\tau} - 1)}{e^{T_j/\tau} - 1} + \lambda (1 - e^{-t/\tau}). \quad (14)$$

From Eq.(14), we obtain the mean value of cycle-time  $T_i$  as the solution of the following equation:

$$\theta = e^{-T_i/\tau} \sum_{j=1}^c w_{ij} \frac{e^{T_j/\tau} (e^{T_i/\tau} - 1)}{e^{T_j/\tau} - 1} + \lambda (1 - e^{-T_i/\tau}). \quad (15)$$

Using the self-consistency Eq.(7), and Eqs.(12) and (15), we obtain the self-consistent condition

$$1 = \left[ \alpha \left( \frac{\alpha}{x} + \bar{\lambda} \right) + \bar{\lambda} \right] (1 - e^{-x}), \quad (16)$$

with the dimensionless parameters  $\bar{\lambda} = \lambda/\theta$ ,  $\alpha = W/\theta$ , and  $x = T/\tau$ , where we redefine the cycle-time as  $T = T_i$ .

The self-consistent condition in the region R2 is obtained similarly as

$$1 = \left[ \alpha' \left( \frac{\alpha'}{x} + \bar{\lambda} \right) + \bar{\lambda} \right] (1 - e^{-x}), \quad (17)$$

where the parameter  $\alpha'$  reflects the construction of synaptic connections  $\{w'_{ij}\}$  in R2, namely  $\alpha' = \sum_j w'_{ij}/\theta$ .

When the synchronized firing occurs between the regions R1 and R2, the firing cycle  $T$  takes the same value in each region. Consequently, we obtain the relation between  $\alpha$  and  $\alpha'$  under synchrony as

$$\bar{\lambda} = \frac{x - \alpha^2(1 - e^{-x})}{x(1 - e^{-x})(\alpha + 1)} = \frac{x - \alpha'^2(1 - e^{-x})}{x(1 - e^{-x})(\alpha' + 1)} \quad (18)$$

The second equation of Eq.(18) has nontrivial solution

$$\alpha' = \frac{\alpha - e^x(\alpha + x)}{(e^x - 1)(\alpha + 1)}. \quad (19)$$

The solutions  $\alpha = \alpha'$  and Eq.(19) are shown in Fig.4. From the discussion in Section 2, in the case of  $|\alpha| > 1$ , R1 shows the spontaneous firing, while, in the case of  $|\alpha| < 1$ , R1 does not show the spontaneous firing. Similar relation is shown in R2 with respect to  $|\alpha'|$ . Then, from Fig.4, it is necessary to lead the nontrivial synchrony that the parameters  $|\alpha|$  and  $|\alpha'|$  take ( $|\alpha| > 1, |\alpha'| > 1$ ), ( $|\alpha| > 1, |\alpha'| < 1$ ) or ( $|\alpha| < 1, |\alpha'| > 1$ ). That is, there exists nontrivial synchrony of two regions only in such cases that at least one region takes conditions of spontaneous firings.

The strength of modulating effects  $\bar{\lambda}$  is shown in Fig.5 using Eq.(18). Especially, when the parameters  $\alpha$  and  $\alpha'$  take the value  $|\alpha| < 1$  and  $|\alpha'| < 1$  (namely both R1 and R2 do not show the spontaneous firing), there exists only trivial relation  $\alpha = \alpha'$  for synchrony. Then, from Eq.(18), the limiting behavior of  $\bar{\lambda}$  for  $x \rightarrow \infty$  and  $x \rightarrow 0$  is obtained as

$$\bar{\lambda}_\infty \equiv \lim_{x \rightarrow \infty} \bar{\lambda} = \frac{1}{1 + \alpha} \quad (20)$$

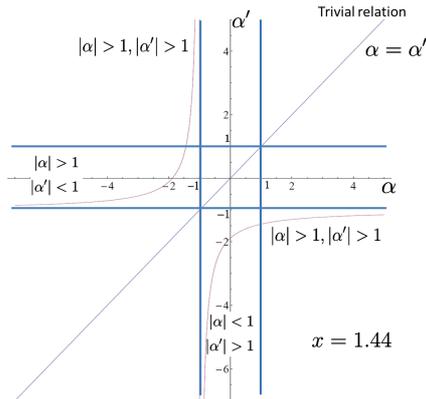


Fig. 4. Relation of  $\alpha$  and  $\alpha'$  where  $x = 1.44$ : The vertical axis express the parameter  $\alpha'$  while the horizontal axis expresses  $\alpha$ . Parameters  $\alpha$  and  $\alpha'$  means  $W/\theta$  and  $W'/\theta$ , respectively. These parameters mean the characters of each region (R1 ( $\alpha$ ) or R2 ( $\alpha'\alpha$  and  $\alpha'$ . We find the non-trivial relation as shown in Eq.(19) only in the cases of ( $|\alpha| > 1, |\alpha'| > 1$ ), ( $|\alpha| > 1, |\alpha'| < 1$ ) and ( $|\alpha| < 1, |\alpha'| > 1$ ).

and

$$\bar{\lambda}_0 \equiv \lim_{x \rightarrow 0} \bar{\lambda} \sim \frac{1 - \alpha}{x}. \quad (21)$$

Consequently, under the condition  $|\alpha|, |\alpha'| < 1$ ,  $\bar{\lambda}$  satisfies the conditions  $\bar{\lambda}_\infty > 0$  and  $\bar{\lambda}_0 \rightarrow +\infty$ , that is, the parameter  $\bar{\lambda}(x)$  takes positive value for all  $x$ . This result shows the following plausible physics: The modulatory effects should enhance the firings of neurons in both regions when the spontaneous firing does not occur in each regions R1 and R2 (Fig.5). Additionally, the R1 and R2 should be constructed by the same configuration (Fig.4). Conditions for synchrony between the regions which do not show the spontaneous firing are very strict.

#### 4 CONCLUSION

We have studied the synchrony firings of neurons in typical two cases.

First, using the effective input theory, we have analyzed the inner region synchrony of neurons (spontaneous firing) using the effective input theory within the region. We have clarified that the inner region synchrony depend on the strength of synaptic connections. When the synaptic connections are weaker than thresholds of neurons, there do not exist spontaneous firings.

Second, we consider the synchrony between two regions connected by modulatory common noises. The synchrony of different regions through the modulators occurs in two typical cases as follows:

- (1) At least one of two regions takes the state for spontaneous firings and receiving appropriate modulation.

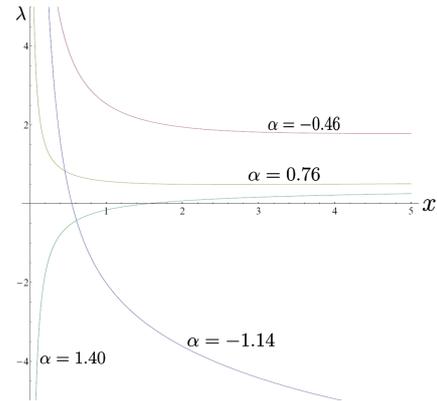


Fig. 5. Dependence of common noise on the synchrony frequency: The vertical axis expresses the parameter  $\bar{\lambda}$  while the horizontal axis expresses the parameter  $x$ . The parameter  $\bar{\lambda}$  means the strength of common noises, and the parameter  $x$  means the cycle-time of a synchrony. We chose the parameters  $\alpha = -1.14, -0.46, 0.76$  and  $1.40$ . These cases are corresponds to each region of Fig.4.

- (2) When both two regions do not take the state for spontaneous firings, these two regions should be constructed by the same synaptic connections for synchrony. Additionally, modulatory effects enhance the firing of neurons in both regions.

The second condition is very strict, and may be very rare.

We have considered the two regions for inter region synchrony. If these regions assumed to be two of many regions in brain, the parameters  $\alpha$  and  $\alpha'$  can be treated as continuous parameters. Then this model of synchrony may change depending on the region of the brain.

This method is generally applicable to other complicated dynamical systems. Although we have used the integrate-and-fire neuron model in this study, any other neuron models can be analyzed generally in the same way.

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