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Classes of linear systems of difference equations with bounded solutions

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Abstract: In this paper we investigate higher order systems of linear difference equations where the associated characteristic matrix polynomial is self-inversive. We consider classes of equations with bounded solutions. It is known that stability properties of higher order systems of linear difference equations are determined by the characteristic values of the corresponding matrix polynomials. All solutions are bounded (in both time directions) if the spectrum of the corresponding matrix polynomial lies on the unit circle, and moreover if the characteristic values of modulus one are semisimple. If the corresponding matrix polynomial is self-inversive then one can use the inner radius of the numerical range to obtain a criterion for boundedness of solutions. We show that all solutions are bounded if the inner radius is greater than 1. In the case of matrix polynomials with positve definite coefficient matrices we derive a computable lower bound for the inner radius.

Keywords: linear difference equations, matrix polynomials, bounded solutions, self-inversive, inner radius, numerical range

1 INTRODUCTION

In this paper we investigate higher order systems of linear difference equations where the associated characteristic matrix polynomial is self-inversive. We consider classes of equations with bounded solutions.

Self-inversive polynomials and matrix polynomials have been studied in the literature under various names including reciprocal, self-reciprocal, palindromic and conjugate-symmetric (see [1], [3], [8], [12]). There are applications in numerous areas of engineering, for example, optimal design of problems governed by hyperbolic field equations [12], the study of line spectral pairs in speech coding [11], and kernel representations of time-reversible systems [9]. Moreover, such polynomials are used in applied mathematics to deal with stability of periodic orbits of autonomous Hamiltonian systems [10], and to investigate Lie algebras for semisimple hypersurface singularities [7].

Only recently self-inversive matrix polynomials and corresponding linear differential and difference equations appeared in the solution of discrete time linear quadratic optimal control problems [2], in the study of discretization schemes for cubic Schrödinger equations [4] and in vibration analysis of railway tracks for high-speed trains [5].

It is known that stability properties of higher order systems of linear difference equations are determined by the characteristic values of the corresponding matrix polynomials.

All solutions are bounded (in both time directions) if the spectrum of the corresponding matrix polynomial lies on the unit circle, and moreover if the characteristic values of modulus one are semisimple (that is the corresponding elementary divisors are linear). If the corresponding matrix polynomial is self-inversive then one can use the inner radius of the numerical range to obtain a criterion for boundedness of solutions. We show that all solutions are bounded if the inner radius is greater than 1. In the case of matrix polynomials with positve definite coefficient matrices we derive a computable lower bound for the inner radius. We illustrate our results by examples.

2 PRELIMINARIES

Let γ be complex number with $|\gamma| = 1$ and $F_0, F_1, \dots, F_m$ complex hermitian $n \times n$ matrices satisfying

$$F_j^* = \gamma F_{m-j},\ j = 0, 1, \dots, m, \qquad \det F_0 \neq 0,\ \det F_m \neq 0. \quad (1)$$

Consider the following higher order systems of linear difference equation:

$$F_m x(t+m) + F_{m-1} x(t+m-1) + \cdots + F_0 x(t) = 0, \quad (2)$$

where $\{x(t)\}_{t=-\infty}^{\infty}$ is a sequence of vectors in $\mathbb{C}^n$ to be determined. The associated characteristic matrix polynomial is as follows:

$$F(z) = F_0 + F_1 z + \cdots + F_m z^m \in \mathbb{C}^{n\times n}[z]. \quad (3)$$

The conjugate-reverse matrix polynomial of $F(z)$ in (3) is defined by

$$\hat{F}(z) = F_m^* + \cdots + F_1^* z^{m-1} + F_0^* z^m.$$

Then it follows form (1) that $F(z) = \gamma \hat{F}(z)$ and thus $F(z)$ is γ-self-inversive.

We use the following notation. If $P(z) \in \mathbb{C}^{n\times n}[z]$ then the set of characteristic values of $P(z)$ is denoted by $\sigma(P) = \{\lambda \in \mathbb{C}; \det P(z) = 0\}$. A characteristic value λ of $P(z)$ is said to be normal if for any $v \in \mathbb{C}^n$,

$$P(\lambda)v = 0 \Longleftrightarrow v^*P(\lambda) = 0.$$

A characteristic value λ of $P(z)$ is said to be semisimple if the corresponding elementary divisors are linear.

Let $W(P) = \{\lambda \in \mathbb{C}; v^*P(\lambda)v = 0 \text{ for some } v \in \mathbb{C}^n, v \neq 0\}$ be the numerical range of $P(z)$. It is obvious that $\sigma(P) \subset W(P)$. We call $r_i(P) = \min\{|\lambda|; \lambda \in W(P)\}$ the inner radius of $P(z)$. If H is hermitian then $\lambda_{\min}(H)$ and $\lambda_{\max}(H)$ shall denote the smallest and the largest eigenvalue of H, respectively. Let $\|H\|$ be the spectral norm of H. Then $\|H\| = \max\{|\lambda_{\min}(H)|, |\lambda_{\max}(H)|\}$ and

$$\|H\| \geq -|\lambda_{\min}(H)|.$$

3 BOUNDEDNESS

In this section we deal with the boundedness for the difference equation (2). The equation (2) is said to be bounded if any solution $x(t)$ of (2) with initial conditions $x(0) = x_0, x(1) = x_1, \ldots, x(m-1) = x_{m-1}$ is bounded for $t \to \infty$ and $t \to -\infty$.

In the rest of this paper we assume that the associated characteristic matrix polynomial $F(z)$ to (2) has the form

$$F(z) = P(z) + \gamma z^r \hat{P}(z) \tag{4}$$

for some $r \geq 0$ and some $P(z) = \sum_{j=0}^{k} A_j z^j \in \mathbb{C}^{n\times n}[z]$. Then $F(z)$ is γ-self-inversive. Note that for any $P(z) = \sum_{j=0}^{k} A_j z^j \in \mathbb{C}^{n\times n}[z]$ and any $r \in \mathbb{Z}, r \geq 0$, $P(z) + \gamma z^r \hat{P}(z)$ is γ-self-inversive.

We have the following fact (see [6]).

Proposition 1. *Let $F(z)$ be a self-inversive matrix polynomial of the form* (4) *and suppose $r_i(P) > 1$. Then the characteristic values of $F(z)$ lie on the unit circle, and they are normal and semisimple.*

An immediate consequence of the preceding proposition is the following.

Theorem 2. *Let $F(z)$ be a self-inversive matrix polynomial of the form* (4) *and suppose $r_i(P) > 1$. Then, the difference equation* (2) *is bounded.*

The following theorem provides a computable lower bound for the inner radius.

Theorem 3. *Let the coefficients A_j, $j = 0, \ldots, k$, of $P(z) = \sum_{j=0}^{k} A_j z^j$ be hermitian and positive definite. Set*

$$\mu(P) = \min\left\{\lambda_{\min}(A_j A_{j+1}^{-1});\ j = 0, \ldots, k-1\right\}.$$

Then $r_i(P) \geq \mu(P)$.

Corollary 4. *If $A_0 > A_1 > \cdots > A_k > 0$ then $\mu(P) > 1$.*

4 ROBUST BOUNDEDNESS

The difference equation (2) is said to be robustly bounded if there exists $\varepsilon > 0$ such that for any hermitian matrices $\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_m$ satisfying

$$\|\tilde{F}_j - F_j\| < \varepsilon,\ F_j^* = \gamma F_{m-j},\ j = 0, 1, \ldots, m,$$

and for any initial conditions $x(0) = x_0, x(1) = x_1, \ldots, x(m-1) = x_{m-1} \in \mathbb{C}^n$, the solution of the difference equation

$$\tilde{F}_m x(t+m) + \tilde{F}_{m-1}x(t+m-1) + \cdots + \tilde{F}_0 x(t) = 0$$

is bounded for $t \to \infty$ and $t \to -\infty$.

First, we consider the following difference equation in the case of $n = 1$:

$$\begin{aligned} a_0 x(t+m) + a_1 x(t+m-1) + \cdots \\ + a_k x(t+m-k) + a_k x(t+k) + \cdots \\ + a_1 x(t+1) + a_0 x(t) = 0,\ m > 2k, \end{aligned} \tag{5}$$

where $a_0, a_1, \ldots, a_k \in \mathbb{R}$ $(a_0 \neq 0)$ are given and $\{x(t)\}_{t=0}^{\infty}$ is a sequence in $\mathbb{R}$ to be determined.

Assume $a_0 > a_1 > \cdots > a_k > 0$. Set

$$\varepsilon = \min\{a_i - a_{i+1} \mid 0 \leq i \leq k-1\}. \tag{6}$$

Suppose that $\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_k \in \mathbb{R}$ satisfy

$$|\tilde{a}_i - a_i| < \frac{1}{2}\varepsilon,\ i = 0, 1, \ldots, k, \tag{7}$$

and consider the difference equation

$$\begin{aligned} \tilde{a}_0 x(t+m) + \tilde{a}_1 x(t+m-1) + \cdots + \\ \tilde{a}_k x(t+m-k) + \tilde{a}_k x(t+k) + \cdots + \\ \tilde{a}_1 x(t+1) + \tilde{a}_0 x(t) = 0 \end{aligned} \tag{8}$$

and the associated characteristic matrix polynomial

$$\begin{aligned} f(z) = \tilde{a}_0 + \tilde{a}_1 z + \cdots + \tilde{a}_k z^k + \\ \tilde{a}_k z^{m-k} + \cdots + \tilde{a}_1 z^{m-1} + \tilde{a}_0 z^m. \end{aligned} \tag{9}$$

The condition (7) implies $\tilde{a}_0 > \tilde{a}_1 > \cdots > \tilde{a}_k$. Thus by Theorem 3.2 in [6] all zeros of f in (9) lie on the unit circle and simple, and hence for any initial conditions $x(0) = x_0, x(1) = x_1, \ldots, x(m-1) = x_{m-1} \in \mathbb{R}$ the solution of the equation (8) is bounded. Therefore the equation (5) is robustly bounded.

Next, we consider the equation (2) with $n \times n$ matrices. Suppose the matrices $M_i = A_{i-1} - A_i$, $i = 1, 2, \ldots, k$, are positive definite. Set $\mu_i = \lambda_{\min}(M_i)$, $i = 1, 2, \ldots, k$ and define $\mu = \min\{\mu_i; i = 1, 2, \ldots, k\}$. Then $M_i \geq \mu I > 0$, $i = 1, 2, \ldots, k$.

Lemma 5. *Suppose* $A_0 > A_1 > \cdots > A_k > 0$. *Let* $\tilde{A}_0, \tilde{A}_1, \ldots, \tilde{A}_k$ *be hermitian* $n \times n$ *matrices satisfying*

$$\|\tilde{A}_i - A_i\| < \frac{\mu}{2},\ i = 0, 1, \ldots, k.$$

Set $\tilde{A}_{k+1} = 0$. *Then*

$$\tilde{A}_{i-1} > \tilde{A}_i,\ i = 0, 1, \ldots, k+1.$$

Proof. Set $\Delta_i = \tilde{A}_i - A_i$. Then

$$\begin{aligned}\tilde{A}_{i-1} - \tilde{A}_i &= (A_{i-1} - A_i) + (\Delta_{i-1} - \Delta_i)\\ &\geq \mu I + (\Delta_{i-1} - \Delta_i).\end{aligned}$$

We have

$$(\Delta_{i-1} - \Delta_i) \geq \lambda_{\min}(\Delta_{i-1} - \Delta_i) I \geq -\|\Delta_{i-1} - \Delta_i\| I.$$

Moreover,

$$\|\Delta_{i-1} - \Delta_i\| \leq \|\Delta_{i-1}\| + \|\Delta_i\| < \mu.$$

Hence

$$(\Delta_{i-1} - \Delta_i) > \mu I,$$

and we obtain $\tilde{A}_{i-1} - \tilde{A}_i > 0$. □

Using the preceding lemma and Proposition 2 we obtain the following result.

Theorem 6. *Let* $F(z)$ *be a self-inversive matrix polynomial of the form* (4) *and suppose that* $A_0, A_1, \ldots, A_k$ *are hermitian with*

$$A_0 > A_1 > \cdots > A_k > 0. \quad (10)$$

Then the difference equation (2) *is robustly bounded.*

5 EXAMPLE

Example 1. Consider the following difference equation with $n = 2$:

$$\begin{aligned}&\begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix} x(t+3) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t+2)\\ &\quad + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x(t+1) + \begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix} x(t) = 0 \quad (11)\end{aligned}$$

The associated characteristic matrix polynomial $F(z)$ is as follows:

$$\begin{aligned}F(z) = &\begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z\\ &+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z^2 + \begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix} z^3 \quad (12)\end{aligned}$$

Setting

$$P(z) = \begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z,$$

$F(z)$ has the form (4) with $\gamma = 1$ and $r = 2$. Thus $F(z)$ is self-inversive. It is easy to see that the coefficients of $F(z)$ are hermitian matrices satisfying

$$\begin{pmatrix} 2 & i \\ -i & 3 \end{pmatrix} > \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} > 0.$$

Therefore, it follows from Theorem 6 that the equation (11) is robustly bounded.

We obtain that the spectrum of $F(z)$ lies on the unit circle and all characteristic values of $F(z)$ are normal and semisimple. In fact, one has

$$\begin{aligned}\det F(z) = (z+1)^2 &\left(z^2 - \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right) z + 1\right)\\ &\left(z^2 - \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) z + 1\right)\end{aligned}$$

and all zeros of $\det F(z)$ lie on the unit circle (see Fig.1).

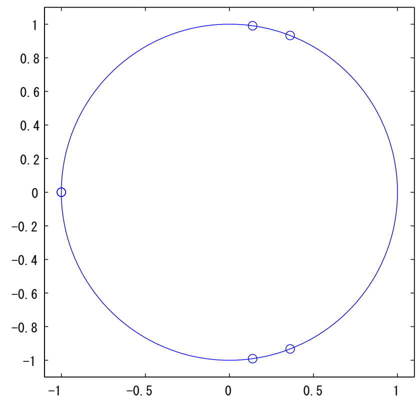

Fig. 1. Characteristic values of (12)

Computing the Smith form $S(z)$ of $F(z)$, we obtain

$$S(z) = \begin{pmatrix} z+1 & 0 \\ 0 & (z+1)p(z) \end{pmatrix}$$

where

$$\begin{aligned}p(z) = &\left(z^2 - \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right) z + 1\right)\\ &\left(z^2 - \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) z + 1\right).\end{aligned}$$

Hence, it can be seen that all characteristic values of $F(z)$ are normal and semisimple.

Example 2. Consider the following diffrence equation with $n = 3$:

$$\begin{aligned}&A_0 x(t+3) + A_1 x(t+2)\\ &\quad + A_1 x(t+1) + A_0 x(t) = 0 \quad (13)\end{aligned}$$

where

$$A_0 = \begin{pmatrix} 2 & i & -i \\ -i & 3 & 0 \\ i & 0 & 3 \end{pmatrix}, \; A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The associated characteristic matrix polynomial $F(z)$ is given by

$$F(z) = P(z) + z^2 \hat{P}(z), \; P(z) = A_0 + A_1 z. \tag{14}$$

Then, it is easy to see that A_0 and A_1 are positive definite hermitian matrices. Moreover, one has $A_0 \not> A_1$, but $A_0 \geq A_1$ because of $\det(zI - (A_0 - A_1)) = z(z-2)(z-3)$.

We compute the inner radius $r_i(P)$. To do so, set $v \in \mathbb{C}, v \neq 0$. Then,

$$v^* P(\lambda) v = \begin{pmatrix} \bar{a} & \bar{b} & \bar{c} \end{pmatrix} P(\lambda) \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \lambda |v|^2 + v^* A_0 v.$$

Thus $r_i(P) = 1$ because of $\det(zI - A_0) = (z-1)(z-3)(z-4)$.

We obtain

$$\begin{aligned} \det F(z) &= 12 + 19z + 27z^2 + 53z^3 + 49z^4 \\ &\qquad + 49z^5 + 53z^6 + 27z^7 + 19z^8 + 12z^9 \\ &= (z+1)^3 (z^2+1) \\ &\qquad (3z^2 - 2z + 3)(4z^2 - 3z + 4) \end{aligned}$$

and thus it can be seen that all zeros of $\det F(z)$ lie on the unit circle (see Fig.2).

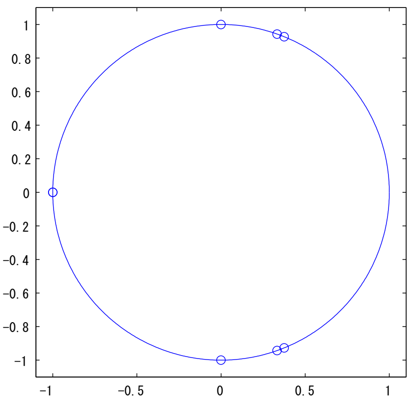

Fig. 2. Characteristic values of (14)

Computing the Smith form $S(z)$ of $F(z)$, we obtain

$$S(z) = \begin{pmatrix} z+1 & 0 & 0 \\ 0 & (z+1) & 0 \\ 0 & 0 & \frac{1}{12}(z+1)p(z) \end{pmatrix}$$

where $p(z) = (z^2+1)(3z^2-2z+3)(4z^2-3z+4)$. Hence, the spectrum of $F(z)$ lies on the unit circle and all characteristic values of $F(z)$ are normal and semisimple. Therefore, (10) is not nessesary for robust boundedness.

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