# Robust Control of a three-link RRR Manipulator with Structured Uncertainty

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**Abstract:** In this paper, we propose a modeling method for an uncertain system of a three-link RRR manipulator. We consider that each rotation joint of this manipulator consists a nominal joint angle and uncertain joint angle. Though, the uncertainty is treated as disturbance in the system that is maximum possible value of the uncertain joint angle. A relationship between disturbance and the system structure in a state equation is clarified. Through the numerical example, we show the effectiveness of our proposed method. It can apply our result to the general method of the robust control with structured uncertainty, such as guaranteed cost control.

Keywords: Linear control system, Robust control, Guaranteed cost control

### **1 INTRODUCTION**

In the framework of the linear control system theory, the design procedure for the controller is archived by a model based method. But the numerical model only contains a nominal characteristic of the plant. Thus, it is important problem to obtain the representation of the effect of the uncertainty, and use this information for the design of robust controller. Chang et al. proposed the design method to guarantee the existence of upper bound of performance index, which called guaranteed cost control (GCC) [1]. Sato et al. consider the trajectory generation problem for energy saving of the manipulator [2, 3]. Authors proposed a modeling method of a linear time invariant system which includes an uncertainty of the plant [4, 5]. And apply the uncertain system to GCC problem. In [6], authors consider GCC problem in the case with the system includes parameter variation in an output matrix. The parameter setting of the free variable i in the linear upper bound is available for adjustment of the closedloop system's characteristic. In [7], authors consider a twolink RR manipulator model case. Such a higher-dimensional system, the effect of uncertainty to the disturbance becomes larger bad influence than smaller dimension one, so that it is not a negligible problem. The observer based method is effective to reduce the effects of disturbance.

In this paper, we apply our modeling method to the threelink RRR manipulator. In such a high dimensional system, the effect of nonlinear element is larger than the lower dimensional system. It is important the robustness of the controller.

#### 2 DERIVATION OF THE UNCERTAIN SYSTEM

The dynamics of a three-link RRR manipulator is illustrated as a following second ordinary differential equation.

$$H(\boldsymbol{\theta})\boldsymbol{\theta} + D\dot{\boldsymbol{\theta}} + \boldsymbol{\eta}(\dot{\boldsymbol{\theta}}, \boldsymbol{\theta}) + g\boldsymbol{\gamma}(\boldsymbol{\theta}) =$$
(1)

where the inertia term is

$$H(\boldsymbol{\theta}) = \begin{bmatrix} h_{11}(\boldsymbol{\theta}) & h_{12}(\boldsymbol{\theta}) & h_{13}(\boldsymbol{\theta}) \\ h_{12}(\boldsymbol{\theta}) & h_{22}(\boldsymbol{\theta}) & h_{23}(\boldsymbol{\theta}) \\ h_{13}(\boldsymbol{\theta}) & h_{23}(\boldsymbol{\theta}) & h_{33}(\boldsymbol{\theta}) \end{bmatrix},$$

the nonlinear term is

$$m{\eta}(m{ heta},\dot{m{ heta}}) ~=~ \left[ egin{array}{c} {}_{11}(\dot{m{ heta}},m{ heta}) \ {}_{21}(\dot{m{ heta}},m{ heta}) ~{}_{22}(\dot{m{ heta}},m{ heta}) \ {}_{31}(\dot{m{ heta}},m{ heta}) ~{}_{32}(\dot{m{ heta}},m{ heta}) \end{array} 
ight],$$

the dumping term is

$$D = \operatorname{diag}(d_1, d_2, d_3),$$

the gravity term and the input term are

$$m{\gamma}(m{ heta}) \ = \ \left[ egin{array}{c} 1(m{ heta}) \ 2(m{ heta}) \ 3(m{ heta}) \end{array} 
ight], \qquad = \ \left[ egin{array}{c} 1 \ 2 \ 2 \ 3 \end{array} 
ight].$$

Here we derive the LTI system with structured uncertainty for the above dynamics. In this system, the relationship between an uncertainty of the system and system structure is expressed in the additional system matrix. Let us assume that the each rotation joints include an angle of the nominal element and an uncertain element. The possible value of the uncertain joint angle is unknown, therefore we use the maximum value of the uncertain angle as a disturbance of the system to derive an uncertain system. The nominal system structure is derived from the nominal joint angle, and the uncertain system structure is derived corresponding to the uncertain joint angle. The system uncertainty is only used to derive the uncertain system, though in the later section of numerical example, the simulation result is calculated by using nominal LTI system.

#### 2.1 Introduce of uncertainty

In this section, we introduce an uncertain angle in the each joint to express the effect to the system structure. Here we consider that the rotation angle  $\bar{i}(t)$  of joint<sub>i</sub>, (i = 1, 2, 3) is constructed from an nominal joint angle i(t) and uncertain joint angle  $\Delta_i$ .

$$\bar{i}_{i} = i + \Delta_{i}, \quad (i = 1, 2, 3)$$
 (2)

where the uncertain joint angle  $\Delta_i$  takes very small value, it can be approximate sinusoidal functions sin, cos as

$$\begin{array}{rcl} \sin \Delta_i & \to & 0 \\ \cos \Delta_i & \to & \Delta c_i \end{array}$$

Then, from the consideration of uncertainty of  $\bar{i}_i$ , from the sinusoidal fundamental formulae, we have

$$\sin \bar{i} = \sin i \cos \Delta i + \cos i \sin \Delta i$$
$$\Delta c_i \sin i \qquad (3)$$

$$\cos \bar{i} = \cos i \cos \Delta i \sin i \sin \Delta i$$
$$\Delta c_i \cos i \qquad (4)$$

In the term of 1 + 2, sin is becomes:

$$\sin(\bar{1} + \bar{2}) = \sin \bar{1} \cos \bar{2} + \cos \bar{1} \sin \bar{2}$$
$$\Delta c_1 \Delta c_2 (\sin \bar{1} \cos \bar{2} + \cos \bar{1} \sin \bar{2})$$
(5)

We consider the approximation of  $\cos$  as follows

$$\cos(\bar{1} + \bar{2}) = \frac{1}{2}(\Delta c_1 + \Delta c_2)\cos(1 + 2)$$
 (6)

Then, the term of  $\bar{}_1(t) + \bar{}_2(t) + \bar{}_3(t)$  becomes

$$\sin(\bar{1}_{1} + \bar{2}_{2} + \bar{3}) \Delta c_{1} \Delta c_{2} \Delta c_{3} (\sin_{1} \cos_{2} + \cos_{1} \sin_{2}) \cos_{3} + \frac{1}{2} (\Delta c_{1} + \Delta c_{2}) \Delta c_{3} \cos(\bar{1}_{1} + 2) \sin_{3}$$
(7)

The Taylor series expansion of Eqs. (3), (4), (5), (6) and (7) near  $_i(t) = 0$ , (i = 1, 2, 3) up to the first-order are express as follows:

$$\begin{array}{cccc} \sin \bar{\phantom{a}}_{1} & \Delta c_{1 \ 1} \\ \cos \bar{\phantom{a}}_{1} & \Delta c_{1} \\ \sin \bar{\phantom{a}}_{2} & \Delta c_{2 \ 2} \end{array}$$

By substituting these approximation terms into each elements  $h_{ij}$ , (i = 1, 2, 3, j = 1, 2, 3) of the inertia term  $H(\theta)$ , The first row elements are:

$$\begin{split} \tilde{h}_{11} &= I_1 + m_1 l_{G1}^2 + I_2 + m_2 (l_1^2 + l_{G2}^2 + 2\Delta c_2 l_1 l_{G2}) \\ &+ I_3 + m_3 (l_1^2 + l_{G2}^2 + l_{G3}^2 + 2\Delta c_2 l_1 l_2 \\ &+ 2\Delta c_3 l_2 l_{G3} + (\Delta c_2 + \Delta c_3) l_1 l_{G3}) \\ \tilde{h}_{12} &= I_2 + m_2 (l_{G2}^2 + \Delta c_2 l_1 l_{G2}) \\ &+ I_3 + m_3 (l_2^2 + l_{G3}^2 + \Delta c_2 l_1 l_2 + 2\Delta c_3 l_2 l_{G3} \\ &+ \frac{1}{2} (\Delta c_2 + \Delta c_3) l_1 l_{G3}) \\ \tilde{h}_{13} &= I_3 + m_3 (l_{G3}^2 + \Delta c_3 l_2 l_{G3} + \frac{1}{2} (\Delta c_2 + \Delta c_3) l_1 l_{G3}) \end{split}$$

The second-row elements are:

$$\begin{split} \tilde{h}_{21} &= \tilde{h}_{12} \\ \tilde{h}_{22} &= I_2 + m_2 l_{G2}^2 + I_3 + m_3 (l_2^2 + l_{G3}^2 + 2\Delta c_3 l_2 l_{G3}) \\ \tilde{h}_{23} &= I_3 + m_3 (l_{G3}^2 + \Delta c_3 l_2 l_{G3}) \end{split}$$

The third-row elements are:

$$\tilde{h}_{31} = \tilde{h}_{13}$$
  
 $\tilde{h}_{32} = \tilde{h}_{23}$ 
  
 $\tilde{h}_{33} = I_3 + m_3 l_{G3}^2$ 

As these results, the inertia term  $H(\boldsymbol{\theta})$  is described as sym-

Table 1. Parameters of the Manipulator

| Parameter | Mean [unit]                                 |  |  |  |  |  |
|-----------|---|--|--|--|--|--|
| $m_i$     | Mass of Link [kg]                           |  |  |  |  |  |
| $I_i$     | Inertia moment of Link [kg·m <sup>2</sup> ] |  |  |  |  |  |
| $l_i$     | length of Link [m]                          |  |  |  |  |  |
| $l_{Gi}$  | Distance from the Joint to the center       |  |  |  |  |  |
|           | of gravity of the Link [m]                  |  |  |  |  |  |
| g         | Gravity [m/sec <sup>2</sup> ]               |  |  |  |  |  |

metric matrix with constant elements.

$$H(\boldsymbol{\theta}) \qquad \tilde{H} = \begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \tilde{h}_{13} \\ \tilde{h}_{21} & \tilde{h}_{22} & \tilde{h}_{23} \\ \tilde{h}_{31} & \tilde{h}_{32} & \tilde{h}_{33} \end{bmatrix}$$
(8)

Next, we shall consider of the linearization of the gravity term. The elements in the first-row are:

$$\tilde{\phantom{a}}_{1} = \Delta c_{1} \Big( m_{1} l_{G1} + m_{2} (l_{1} + \Delta c_{2} l_{G2}) \\ + m_{3} (l_{1} + \Delta c_{2} l_{2} + \Delta c_{2} \Delta c_{3} l_{G3}) \Big)_{1} \\ \Delta c_{1} \Delta c_{2} \Big( m_{2} l_{G2} + m_{3} (l_{2} + \Delta c_{3} l_{G3}) \Big)_{2} \\ \frac{1}{2} (\Delta c_{1} + \Delta c_{2}) \Delta c_{3} m_{3} l_{G3} \Big|_{3} \\ = \Big[ \tilde{\phantom{a}}_{11} \tilde{\phantom{a}}_{12} \tilde{\phantom{a}}_{13} \Big] \theta$$
(9)

where

$$\tilde{\phantom{a}}_{11} = \Delta c_1 (m_1 l_{G1} + m_2 (l_1 + \Delta c_2 l_{G2}) \\
+ m_3 (l_1 + \Delta c_2 l_2 + \Delta c_2 \Delta c_3 l_{G3}))$$

$$\tilde{\phantom{a}}_{12} = \Delta c_1 \Delta c_2 (m_2 l_{G2} + m_3 (l_2 + \Delta c_3 l_{G3})) \\
\tilde{\phantom{a}}_{13} = \frac{1}{2} (\Delta c_1 + \Delta c_2) \Delta c_3 m_3 l_{G3}$$

The second-row elements are:

$$\tilde{\phantom{a}}_{2} = \Delta c_{1} \Delta c_{2} \left( (m_{2}l_{G2} + m_{3}l_{2}) + \Delta c_{3}m_{3}l_{G3} \right)_{1} \\ \Delta c_{1} \Delta c_{2} \left( (m_{2}l_{G2} + m_{3}l_{2}) + \Delta c_{3}m_{3}l_{G3} \right)_{2} \\ \frac{1}{2} (\Delta c_{1} + \Delta c_{2}) \Delta c_{3}m_{3}l_{G3}_{3} \\ = \left[ \tilde{\phantom{a}}_{21} \tilde{\phantom{a}}_{22} \tilde{\phantom{a}}_{23} \right] \boldsymbol{\theta}$$
(10)

where

$$\tilde{\phantom{a}}_{21} = \Delta c_1 \Delta c_2 (m_2 l_{G2} + m_3 (l_2 + \Delta c_3 l_{G3}))$$

$$\tilde{\phantom{a}}_{22} = \Delta c_1 \Delta c_2 (m_2 l_{G2} + m_3 (l_2 + \Delta c_3 l_{G3}))$$

$$\tilde{\phantom{a}}_{23} = \frac{1}{2} (\Delta c_1 + \Delta c_2) \Delta c_3 m_3 l_{G3}$$

The third-row elements are:

$$\tilde{\phantom{a}}_{3} = \Delta c_{1} \Delta c_{2} \Delta c_{3} m_{3} l_{G3 \ 1}$$

$$\Delta c_{1} \Delta c_{2} \Delta c_{3} m_{3} l_{G3 \ 2}$$

$$\frac{1}{2} (\Delta c_{1} + \Delta c_{2}) \Delta c_{3} m_{3} l_{G3 \ 3}$$

$$= \left[ \tilde{\phantom{a}}_{31} \tilde{\phantom{a}}_{32} \tilde{\phantom{a}}_{33} \right] \boldsymbol{\theta} \qquad (11)$$

where

$$\tilde{a}_{31} = \Delta c_1 \Delta c_2 \Delta c_3 m_3 l_{G3}$$

$$\tilde{a}_{32} = \Delta c_1 \Delta c_2 \Delta c_3 m_3 l_{G3} = \tilde{a}_{31}$$

$$\tilde{a}_{33} = \frac{1}{2} (\Delta c_1 + \Delta c_2) \Delta c_3 m_3 l_{G3} = \tilde{a}_{13}$$

From Eqs. (9), (10) and (11), the linearized gravity term is obtained as:

$$g\boldsymbol{\gamma}(\boldsymbol{\theta})$$
  $\tilde{\Gamma}\boldsymbol{\theta}(t), \ \tilde{\Gamma} = g \begin{bmatrix} \tilde{1}1 & 12 & 13\\ \tilde{2}1 & 22 & 23\\ \tilde{3}1 & 32 & 33 \end{bmatrix}$ 

#### 2.2 Formulation of the uncertain system

Here we us assume that the derivative of the angle i takes vary small value, the product term becomes  $i j \rightarrow 0$ , (i = 1, 2, 3, j = 1, 2, 3). Thus, it can ignore the nonlinear term in Eq. (1). Let  $\eta(\theta, \dot{\theta}) = 0$ 

$$H(\boldsymbol{\theta})\boldsymbol{\theta}(t) + D\dot{\boldsymbol{\theta}}(t) + g\boldsymbol{\gamma}(\boldsymbol{\theta}) =$$
(12)

Let us substitute Eqs. (8), (12), then

$$\tilde{H}\boldsymbol{\theta}(t) + D\dot{\boldsymbol{\theta}}(t) + \tilde{\Gamma}\boldsymbol{\theta}(t) =$$
(13)

By multiplying  $\tilde{H}^{-1}$  from the left side to Eq.(13),

$$\boldsymbol{\theta}(t) = \tilde{H}^{-1} D \dot{\boldsymbol{\theta}}(t) \quad \tilde{H}^{-1} \tilde{\Gamma} \boldsymbol{\theta}(t) + \tilde{H}^{-1}$$
(14)

From the above equation, we can compose the augmented system as:

By the definition of the state vector  $\boldsymbol{x}(t)$  and the input vector  $\boldsymbol{u}(t)$  as:

$$\boldsymbol{x}(t) = \begin{array}{c} \boldsymbol{\theta}(t) \\ \dot{\boldsymbol{\theta}}(t) \end{array} = \begin{bmatrix} 1(t) \\ 2(t) \\ 3(t) \\ .1(t) \\ .2(t) \\ .3(t) \end{bmatrix}, \boldsymbol{u}(t) = \begin{bmatrix} 1(t) \\ 2(t) \\ .3(t) \end{bmatrix}$$

We can obtain the uncertain LTI system.

$$\dot{\boldsymbol{x}}(t) = A(\boldsymbol{\xi})\boldsymbol{x}(t) + B(\boldsymbol{\zeta})\boldsymbol{u}(t)$$

where state matrix A() and intput matrix B() are:

$$A(\boldsymbol{\xi}) = \begin{array}{cc} \mathbf{O} & \mathbf{I} \\ \tilde{H}^{-1}\tilde{\boldsymbol{\Gamma}} & \tilde{H}^{-1}\boldsymbol{D} \end{array}, B(\boldsymbol{\zeta}) = \begin{array}{cc} \mathbf{O} \\ \tilde{H}^{-1} \end{array}$$

The uncertain system is express as  $A(\boldsymbol{\xi})$  and  $B(\boldsymbol{\zeta})$ . This matrix is separates to a nominal elements  $A_0, B_0$  and an uncertain elements  $\Delta A, \Delta B$ .

$$A(\boldsymbol{\xi}) = A_0 + \Delta A \tag{15}$$

$$B(\boldsymbol{\zeta}) = B_0 + \Delta B \tag{16}$$

The structure of the nominal elements is express as  $A_0, B_0$ . The structure of the uncertain elements  $A_i$  and  $B_i$  is defined by uncertainty  $\Delta_i$  where i = 1, 2, 3.

$$\Delta A = \sum_{i=1}^{3} {}_{i}A_{i}, | {}_{i}| 1$$
 (17)

$$\Delta B = \sum_{j=1}^{3} \zeta_j B_j, \ |\zeta_j| = 1$$
(18)

where  $_i$  and  $\zeta_j$  are scalar values which express a scale of uncertainty.  $A_i$  and  $B_j$  are matrices which expressed a structure of the uncertainty.

#### **3 NUMERICAL RESULT**

In this section, we show the numerical example that is modeling method of the state and input matrix with structured uncertainty in the state equation. Each parameters takes as in the table 2. The nominal system structure is obtained as

Table 2. Parameters (i = 1, 2, 3)

| Parameter    | value | Parameter | value |
|--------------|-------|-----------|-------|
| $m_i$        | 1.00  | $l_{Gi}$  | 0.15  |
| $I_i$        | 0.30  | g         | 9.80  |
| $\Delta c_i$ | 0.03  | $d_i$     | 0.03  |

follows:

| $A_0$ | =             |          |           |      |         |  |
|-------|---------------|----------|-----------|------|---------|--|
| Γ     | 0.00          | 0.00     | 0.00      | 1.00 | 0.00    | 0.00   |
|       | 0.00          | 0.00     | 0.00      | 0.00 | 1.00    | 0.00   |
|       | 0.00          | 0.00     | 0.00      | 0.00 | 0.00    | 1.00   |
|       | 74.34         | 81.94    | 14.42     | 0.64 | 1.10    | 0.16   |
|       | 104.35        | 164.46   | 24.81     | 1.10 | 2.05    | 0.44   |
| L     | 10.77         | 29.62    | 11.35     | 0.16 | 0.44    | 0.51   |
|       |               |          |           |      |         |  |
| B     | $b_0 = [0.0]$ | 0 0.00 0 | 0.00 21.2 | 6 36 | .57 5.5 | $\begin{bmatrix} 0 \end{bmatrix}^{\mathrm{T}}$ |

State matrix and input matrix of the structured uncertainty are:

| A | $A_1 = 10$ | 7    |      |      |      |      |
|---|------------|------|------|------|------|------|
| ſ | 0.00       | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 0.00       | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 0.00       | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 3.72       | 4.09 | 0.36 | 0.00 | 0.00 | 0.00 |
| 1 | 5.22       | 8.22 | 0.62 | 0.00 | 0.00 | 0.00 |
|   | 0.54       | 1.48 | 0.28 | 0.00 | 0.00 | 0.00 |

$$A_2 = 10^{-7}$$

| - | 0.00   | 0.00  | 0.00 | 0.00 | 0.00 | 0.00 |
|---|--------|-------|------|------|------|------|
|   | 0.00   | 0.00  | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 0.00   | 0.00  | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 9.1726 | 26.54 | 4.02 | 0.15 | 0.27 | 0.06 |
|   | 17.277 | 49.84 | 7.44 | 0.27 | 0.51 | 0.10 |
| _ | 3.7671 | 9.30  | 1.02 | 0.05 | 0.10 | 0.02 |
| _ | 3.7671 | 9.30  | 1.02 | 0.05 | 0.10 | 0.02 |

| A | $_{3} = 10$ | 7    |      |      |      |      |
|---|-------------|------|------|------|------|------|
| Γ | 0.00        | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 0.00        | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 0.00        | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
|   | 0.90        | 1.12 | 0.56 | 0.01 | 0.02 | 0.01 |
|   | 1.91        | 2.92 | 0.84 | 0.02 | 0.05 | 0.02 |
|   | 0.25        | 1.76 | 0.40 | 0.01 | 0.02 | 0.01 |

| $B_1$ | = | $0_{6\times}$ | 1 |               |      |      |      |      |                    |
|-------|---|---------------|---|---------------|------|------|------|------|--------------------|
| $B_2$ | = | 10            | 7 | $\big[\ 0.00$ | 0.00 | 0.00 | 4.86 | 9.13 | 1.78] <sup>T</sup> |
| $B_3$ | = | 10            | 7 | [ 0.00        | 0.00 | 0.00 | 0.27 | 0.66 | 0.27] <sup>T</sup> |

## 4 CONCLUSION

In this paper, we proposed the design method of uncertain system of three-link RRR manipulator with parameter variation in each joint angle. The relationship between an uncertainty and system structure is clarified in the structured uncertainty in the state matrix and input matrix of the state differential equation. The numerical example will be shown that the effectiveness of our method. Future study is to apply GCC to our method to design a robust controller.

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