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An application of guaranteed cost control to a 3-DOF model helicopter

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Abstract: This paper deals with guaranteed cost control for a model helicopter which has 3-DOF (degree of freedom): the elevation, pitch, and travel angles. One of main difficulties in designing a feedback controller for the helicopter is that the model includes nonlinearities. In this paper, these nonlinearities are considered as the uncertainty terms. Guaranteed cost control is applied not only to achieve the closed-loop stability but also to guarantee an adequate level of performance of the nonlinear 3-DOF model helicopter. A numerical example is shown to illustrate the effectiveness of the proposed method.

Keywords: guaranteed cost control, 3-DOF model helicopter, LMI

1 INTRODUCTION

This paper deals with guaranteed cost control for a model helicopter which has 3-DOF (degree of freedom): the elevation, pitch, and travel angles [1], [2]. One of main difficulties in designing a feedback controller for the helicopter is that the dynamics include nonlinearities.

Although single-input single-output approaches have advantages in simple structure, straightforward and so on, these methods are difficult to consider uncertainties. Therefore, development of multi-input multi-output control approaches are widely applied, see e.g. [3], [4]. Moreover, avoiding difficulties in measurement of system state due to uncertainties, an observer can be applied to reconstruct the system dynamics [5].

In this paper, these nonlinearities are considered as the uncertainty terms. By using Taylor's expansion, the state equation of a nonlinear 3-DOF model helicopter is changed to the form of a continuous-time uncertain system. Because the presence of the uncertainties may cause instability and bad performance on a controlled system, then guaranteed cost control method is applied.

The objective of this paper is to propose a design method of guaranteed cost control with a minimal order observer for a 3-DOF model helicopter via linear matrix inequalities (LMIs) feasible solutions.

Finally, a numerical example is given to illustrate the effectiveness of the proposed method and it is shown that a 3-DOF nonlinear model helicopter can be stabilized by the guaranteed cost control method.

2 MODEL HELICOPTER

The dynamics of a 3-DOF model helicopter shown in Fig. 1. are described [1] as

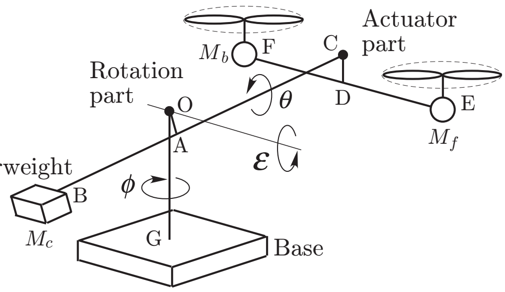

Fig. 1. A 3-DOF model helicopter

$$\dot{\boldsymbol{x}}_p = \begin{bmatrix} p_1\cos\varepsilon + p_2\sin\varepsilon + p_3\dot{\varepsilon} + p_4\cos\theta v_1 \\ p_5\cos\theta + p_6\sin\theta + p_7\dot{\theta} + p_8 v_2 \\ p_9\dot{\phi} + p_{10}\sin\theta v_1 \\ \dot{\varepsilon} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \quad (1)$$

where p_i, (i=1,...,10) are model helicopter constants; ε, θ, ϕ are the elevation, pitch and travel angles and

$$\boldsymbol{x}_p = \quad \dot{\varepsilon}, \dot{\theta}, \dot{\phi}, \varepsilon, \theta, \phi \quad ^T,$$
$$v_1 = V_f + V_b, \; v_2 = V_f - V_b,$$

V_f and V_b are voltages applied to the front and rear motor, respectively.

3 PROBLEM STATEMENT

By using Taylor's expansion, a 3-DOF nonlinear model helicopter (1) can be expressed by the form

$$\dot{\boldsymbol{x}}(t) = (A + \Delta A(t))\boldsymbol{x}(t) + (B + \Delta B(t))\boldsymbol{u}(t) \quad (2)$$
$$\boldsymbol{y}(t) = C\boldsymbol{x}(t), \; C = [O \;\; I_3] \quad (3)$$

$$\boldsymbol{u}(t) = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},\ u_1 = v_1 + \frac{p_1}{p_4},\ u_2 = v_2 + \frac{p_5}{p_8} \tag{4}$$

where

$$\begin{aligned}
A &= [a_{ij}],\ (i,j = 1,...,6),\\
a_{11} &= p_3,\ a_{14} = p_2,\ a_{22} = p_7,\ a_{25} = p_6,\\
a_{33} &= -\frac{p_1 p_{10}}{p_4},\ a_{41} = a_{52} = a_{63} = 1,\\
B &= [b_{ij}],\ (i = 1,...,6,\ j = 1,2),\\
b_{11} &= p_4,\ b_{21} = p_8,\\
\Delta A &= [\delta a_{ij}],\ (i,j = 1,...,6),\\
\delta a_{11} &= \Delta A_{11},\ \delta a_{14} = \Delta A_{14},\ \delta a_{15} = \Delta A_{15},\\
\delta a_{22} &= \Delta A_{22},\ \delta a_{25} = \Delta A_{25},\ \delta a_{33} = \Delta A_{33},\\
\delta a_{35} &= \Delta A_{35},\\
\Delta B &= [\delta b_{ij}],\ (i = 1,...,6,\ j = 1,2),\\
\delta b_{11} &= \Delta B_{11},\ \delta b_{22} = \Delta B_{22},\ \delta b_{31} = \Delta B_{31},
\end{aligned}$$

$$\begin{aligned}
\Delta A_{11} &= \Delta p_3,\\
\Delta A_{14} &= p_2(-\frac{1}{3!}x_4^2 + O(x_4^4)) + p_1(-\frac{1}{2}x_4 + O(x_4^3)) + \Delta p_2,\\
\Delta A_{15} &= p_1(-\frac{1}{2}x_5 + O(x_5^3)),\ \Delta A_{22} = \Delta p_7,\\
\Delta A_{25} &= p_5(-\frac{1}{2}x_5 + O(x_5^3)) + p_6(-\frac{1}{3!}x_6^2 + O(x_6^4)) + \Delta p_6,\\
\Delta A_{33} &= \Delta p_9,\\
\Delta A_{35} &= -\frac{p_1 p_{10}}{p_4}(-\frac{1}{3!}x_5^2 + O(x_5^4)) - \Delta(p_1 p_{10}/p_4),\\
\Delta B_{11} &= p_4(-\frac{1}{2}x_5 + O(x_5^4)) + \Delta p_4,\\
\Delta B_{22} &= \Delta p_8,\ \Delta B_{31} = p_{10}(x_5 - \frac{1}{3!}x_5^3 + O(x_5^5)),
\end{aligned}$$

while other elements are zero, and $\boldsymbol{x}(t) \in \Re^n$ is the state vector, $\boldsymbol{u}(t) \in \Re^r$ is the control input vector, $\boldsymbol{y}(t) \in \Re^m$ is the measured output vector, Δp_i denote uncertain terms of p_i, A, B, C are known constant real-valued matrices with appropriate dimensions. Due to the system constraint, we have the bounds of $|\varepsilon|$, $|\theta|$ and $|\phi|$ as $\varepsilon_{max}(= 0.3)$, $\theta_{max}(= 0.3)$ and $\phi_{max}(= 0.8)$.

Under the limitations of $|\varepsilon|$, $|\theta|$ and $|\phi|$, matrices $\Delta A(t)$ and $\Delta B(t)$ can be represented as

$$\Delta A(t) = D_A F_A(t) E_A,\ \Delta B(t) = D_B F_B(t) E_B \tag{5}$$

with

$$F_A^T(t) F_A(t) \le I,\ F_B^T(t) F_B(t) \le I$$

where D_A, D_B, E_A, E_B are constant real-valued known matrices with appropriate dimensions, and $F_A(t)$ and $F_B(t)$ are real time-varying unknown continuous and deterministic matrices.

We assume that the initial state variable $\boldsymbol{x}(0)$ is unknown, but their mean and covariance are known

$$E[\boldsymbol{x}(0)] = \boldsymbol{m}_0 \tag{6}$$

$$E\ \ (\boldsymbol{x}(0) - \boldsymbol{m}_0)(\boldsymbol{x}(0) - \boldsymbol{m}_0)^T\ \ = \Sigma_0 > O \tag{7}$$

where $E[\cdot]$ denotes the expected value operator.

The problem considered here is to design a guaranteed cost controller with a minimal order observer so as to achieve an upper bound on the following quadratic performance index

$$E[J] = E\left[\int_0^\infty (\boldsymbol{x}^T(t) Q \boldsymbol{x}(t) + \boldsymbol{u}^T(t) R \boldsymbol{u}(t)) dt\right] < E[J^*] \tag{8}$$

associated with the uncertain system (2) where Q and R are given symmetric positive-definite matrices.

4 GUARANTEED COST CONTROLLER DESIGN

Design of a guaranteed cost controller is described in the following equations. Here, a minimal order observer is given by

$$\dot{\boldsymbol{z}}(t) = D\boldsymbol{z}(t) + E\boldsymbol{y}(t) + F\boldsymbol{u}(t) \tag{9}$$

$$\hat{\boldsymbol{x}}(t) = P\boldsymbol{z}(t) + W\boldsymbol{y}(t) \tag{10}$$

with

$$\begin{aligned}
&D = A_{11} + LA_{21},\ PT + WC = I_6,\\
&F = TB,\ TA - DT = EC,\ A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right],\\
&P = \ \ I_3\ \boldsymbol{0}\ \ ^T,\ T = \ \ I_3\ L
\end{aligned}$$

and a controller is assumed to have a form of

$$\boldsymbol{u}(t) = K\hat{\boldsymbol{x}}(t),\ K = -R^{-1}B^T S_1 \tag{11}$$

where S_1 is a symmetric positive definite matrix.

Then, the following Theorem gives a design method of guaranteed cost control to the 3-DOF model helicopter (2)-(3).

Theorem 1. If the following matrix inequalities optimization problem; $\min\{\gamma_0 + \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4\}$ subject to

$$\begin{bmatrix} \Lambda_0 & X^T & XE_A & XE_A \\ * & -Q^{-1} & 0 & 0 \\ * & 0 & -\alpha_1 I & 0 \\ * & 0 & 0 & -\alpha_4 I \end{bmatrix} < 0 \tag{12}$$

$$\begin{bmatrix} \bar{\Lambda}_0 & G_1 & G_2 & G_2 & G_3 & G_3 & G_4 \\ * & -R & 0 & 0 & 0 & 0 & 0 \\ * & 0 & -\alpha_{5,inv}I & 0 & 0 & 0 & 0 \\ * & 0 & 0 & -\alpha_{6,inv}I & 0 & 0 & 0 \\ * & 0 & 0 & 0 & -\alpha_3 I & 0 & 0 \\ * & 0 & 0 & 0 & 0 & -\alpha_6 I & 0 \\ * & 0 & 0 & 0 & 0 & 0 & -\alpha_{4,inv}I \end{bmatrix} < 0 \tag{13}$$

$$\sum_{k=1}^{6} \boldsymbol{e}_{6k}^T \Theta_0 \boldsymbol{e}_{6k} < \gamma_0,\ \sum_{k=1}^{3} \boldsymbol{e}_{3k}^T \Theta_1 \boldsymbol{e}_{3k} < \gamma_1,$$
$$\sum_{k=1}^{3} \boldsymbol{e}_{3k}^T \Theta_2 \boldsymbol{e}_{3k} < \gamma_2,\ \sum_{k=1}^{3} \boldsymbol{e}_{3k}^T \Theta_3 \boldsymbol{e}_{3k} < \gamma_3 \tag{14}$$

$$\begin{bmatrix} -\gamma_4 & \boldsymbol{v}_1^T Y^T & \boldsymbol{v}_2^T Y^T & \boldsymbol{v}_3^T Y^T \\ Y\boldsymbol{v}_1 & -S_2 & & \\ Y\boldsymbol{v}_2 & & -S_2 & \\ Y\boldsymbol{v}_3 & & & -S_2 \end{bmatrix} < 0 \tag{15}$$

where

$$\begin{aligned} \Lambda_0 &= AX + XA^T - BR^{-1}B^T + (\alpha_2 + \alpha_3)D_B D_B^T \\ &\quad + \alpha_1 D_A D_A^T + (\alpha_{2,inv} + \alpha_{5,inv})BR^{-1}E_B^T E_B R^{-1} B^T, \\ \bar{\Lambda}_0 &= S_2 A_{11} + A_{11}^T S_2 + Y A_{21} + A_{21}^T Y^T, \\ Y &= S_2 L,\ Z = [S_2\ \ Y],\ G_1 = P^T S_1 B, \\ G_2 &= Z D_B,\ G_3 = P^T S_1 B R^{-1} E_B^T,\ G_4 = Z D_A, \\ \Theta_0 &= \frac{1}{2}(S_1(\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T) + (\Sigma_0 + \boldsymbol{m}_0 \boldsymbol{m}_0^T)^T S_1), \\ \Theta_1 &= \frac{1}{2}(S_2 \Sigma_{11} + \Sigma_{11} S_2),\ \Theta_2 = \frac{1}{2}(Y\Sigma_{21} + \Sigma_{21}^T Y^T), \\ \Theta_3 &= \frac{1}{2}(Y^T \Sigma_{12} + \Sigma_{12}^T Y),\ \boldsymbol{e}_{ik} = \ \boldsymbol{0}_{k-1}^T\ 1\ \boldsymbol{0}_{i-k}^T\ ^T, \\ \Sigma_0 &= \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix},\ \Sigma_{22}^{1/2} = [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3] \end{aligned}$$

has a solution $S_1 > 0$, $S_2 > 0$, $X > 0$, Y, Z, $\alpha_1 > 0$, $\alpha_2 > 0$, $\alpha_{2,inv} > 0$, $\alpha_3 > 0$, $\alpha_4 > 0$, $\alpha_{4,inv} > 0$, $\alpha_{5,inv} > 0$, $\alpha_6 > 0$, $\alpha_{6,inv} > 0$, γ_0, γ_1, γ_2, γ_3, γ_4 which satisfy the relation $\alpha_2^{-1} = \alpha_{2,inv}$, $\alpha_4^{-1} = \alpha_{4,inv}$, $\alpha_6^{-1} = \alpha_{6,inv}$ and $S_1^{-1} = X$, then the minimal order observer-based control law (9)-(11) is a guaranteed cost controller which gives the minimum expected value of the guaranteed cost

$$E[J^*] = E\left[\boldsymbol{x}^T(0) S_1 \boldsymbol{x}(0) + \boldsymbol{\xi}^T(0) S_2 \boldsymbol{\xi}(0)\right] \tag{16}$$

where $\boldsymbol{\xi}(t) = \boldsymbol{z}(t) - T\boldsymbol{x}(t)$ is the estimated error of the minimal order observer.

Remark 1: Since inequalities in (12) and (13) contain scalars and matrices that satisfy inverse relations $S_1^{-1} = X$, $\alpha_2^{-1} = \alpha_{2,inv}$, $\alpha_4^{-1} = \alpha_{4,inv}$, and $\alpha_6^{-1} = \alpha_{6,inv}$. then an iterative LMI algorithm is adopted to solve [6],[7].

5 A NUMERICAL EXAMPLE

The nominal values of the model helicopter are as follows:

$$\begin{aligned} p_1 &= [-(M_f + M_b)gL_a + M_c g L_c]/J_\varepsilon, \\ p_2 &= [-(M_f + M_b)gL_a \tan\delta_a + M_c g L_c \tan\delta_c]/J_\varepsilon, \\ p_3 &= \eta_\varepsilon/J_\varepsilon,\ p_4 = K_m L_a/J_\varepsilon,\ p_5 = (-M_f + M_b)gL_h/J_\theta, \\ p_6 &= -(M_f + M_b)gL_h \tan\delta_h/J_\theta,\ p_7 = -\eta_\theta/J_\theta, \\ p_8 &= K_m L_h/J_\theta,\ \ p_9 = -\eta_\phi/J_\phi, p_{10} = -K_m L_a/J_\phi, \\ \delta_a &= \tan^{-1}[(L_d + L_e)/L_a],\ \delta_c = \tan^{-1}(L_d/L_c), \\ \delta_h &= \tan^{-1}(L_e/L_h),\ J_\varepsilon = 0.86 \text{ kg m}^2, J_\theta = 0.044 \text{ kg m}^2, \\ J_\phi &= 0.82 \text{ kg m}^2, L_a = 0.62 \text{ m},\ L_c = 0.44 \text{ m}, \\ L_d &= 0.05 \text{ m},\ L_e = 0.02 \text{ m},\ L_h = 0.177 \text{ m}, \\ M_f &= 0.69 \text{ kg}, M_b = 0.69 \text{ kg},\ M_c = 1.67 \text{ kg}, \\ K_m &= 0.5 \text{ N/V},\ g = 9.81 \text{ m/s}^2, \\ \eta_\varepsilon &= 0.001 \text{ kg m}^2\text{/s},\ \eta_\theta = 0.001 \text{ kg m}^2\text{/s}, \\ \eta_\phi &= 0.005 \text{ kg m}^2\text{/s}, \end{aligned}$$

and the uncertain parameters Δp_2, Δp_3, Δp_4, Δp_5, Δp_6, Δp_7, Δp_8, Δp_9 $\Delta(p_1 p_{10}/p_4)$ are 5% of each p_2,p_3, p_4, p_5, p_6, p_7, p_8, p_9 and $(p_1 p_{10}/p_4)$, respectively. Next, D_A, E_A, D_B, E_B, $\boldsymbol{m}_0$, Σ_0, R, Q are given as

$$\begin{aligned} D_A &= \ d_{A_{ij}}\ \ , (i,j = 1,...,6) \\ d_{A_{11}} &= \sqrt{|\ \Delta A_{11}\ |}, d_{A_{14}} = \sqrt{|\ \Delta A_{14}\ |}, \\ d_{A_{15}} &= \sqrt{|\ \Delta A_{15}\ |}, d_{A_{22}} = \sqrt{|\ \Delta A_{22}\ |}, \\ d_{A_{26}} &= -\sqrt{|\ \Delta A_{25}\ |}, d_{A_{33}} = -\sqrt{|\ \Delta A_{35}\ |}, \\ F_A(t) &= I_6,\ E_A = \ e_{A_{ij}}\ \ , (i,j = 1,...,6) \\ e_{A_{11}} &= \sqrt{|\ \Delta A_{11}\ |}, e_{A_{22}} = \sqrt{|\ \Delta A_{22}\ |}, \\ e_{A_{33}} &= \frac{\Delta A_{33}}{\sqrt{|\ \Delta A_{35}\ |}}, e_{A_{35}} = \sqrt{|\ \Delta A_{35}\ |}, \\ e_{A_{44}} &= \sqrt{|\ \Delta A_{14}\ |}, e_{A_{55}} = \sqrt{|\ \Delta A_{15}\ |}, \\ e_{A_{65}} &= \sqrt{|\ \Delta A_{25}\ |} \\ D_B &= \ d_{B_{ij}}\ \ , (i = 1,...,6; j = 1,2) \\ d_{B_{12}} &= \frac{-\Delta B_{11}}{\sqrt{|\ \Delta B_{31}\ |}}, d_{B_{21}} = \sqrt{|\ \Delta B_{22}\ |}, \\ d_{B_{22}} &= -\frac{\sqrt{|\ \Delta B_{22}\ |} \times \sqrt{|\ \Delta B_{11}\ |}}{\sqrt{|\ \Delta B_{31}\ |}}, d_{B_{32}} = \sqrt{|\ \Delta B_{31}\ |} \\ F_B(t) &= I_2,\ E_B = \begin{bmatrix} -\sqrt{|\ \Delta B_{11}\ |} & \sqrt{|\ \Delta B_{22}\ |} \\ -\sqrt{|\ \Delta B_{31}\ |} & 0 \end{bmatrix}, \\ \boldsymbol{m}_0 &= \boldsymbol{0}_6,\ \Sigma_0 = 0.036 I_6, R = I_2, \\ Q &= \text{diag}(0.1, 0.1, 0.1, 1, 1, 1), \end{aligned}$$

while other elements of $d_{A_{ij}}$, $e_{A_{ij}}$, and $d_{B_{ij}}$ are zero.

Results of the controller gain K, the observer gain L and the expected guaranteed cost $E[J^*]$ are obtained below

$$K = \begin{bmatrix} -2.4649 & 0.0078 & 0.0112 & -0.1667 & -0.0467 & -0.0033 \\ 0.0433 & -2.4787 & 7.2135 & -0.1399 & -3.0527 & 2.3029 \end{bmatrix},$$

$$L = \begin{bmatrix} -2.9302 & 0.0574 & -0.1278 \\ 0.0574 & -2.3582 & 2.6908 \\ -0.1278 & 2.6908 & -7.6101 \end{bmatrix}, \ E[J^*] = 28.9098.$$

Figure 2 shows the transition of the guaranteed cost, and Fig. 3-5 show the trajectories of elevation, pitch and travel angles with $\boldsymbol{x}(0) = [0\ 0\ 0\ 0.2\ 0.2\ 0.1]^T$.

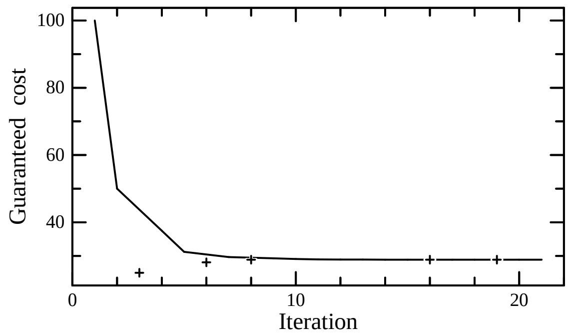

Fig. 2. Transition of the guaranteed cost

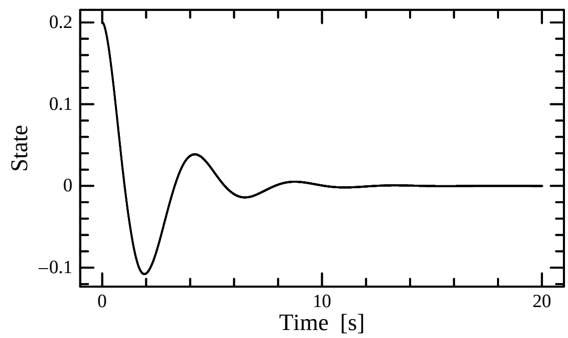

Fig. 3. Trajectory of elevation angle ε

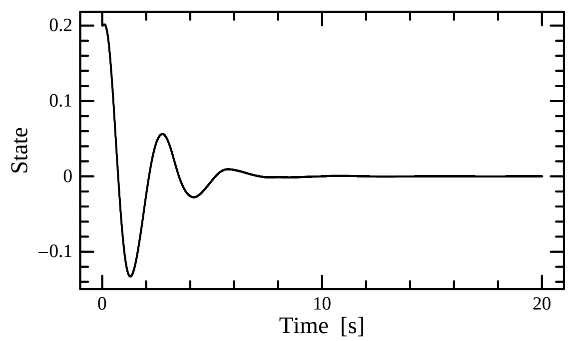

Fig. 4. Trajectory of pitch angle θ

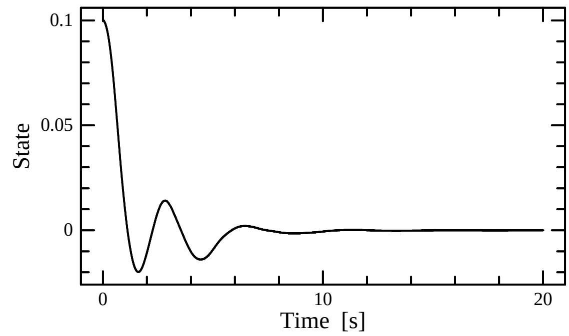

Fig. 5. Trajectory of travel angle ϕ

6 CONCLUSION

This paper discusses a design method of guaranteed cost control with a minimal order observer for a 3-DOF nonlinear model helicopter via linear matrix inequalities (LMIs). The results show the effectiveness of the proposed method.

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