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Sampled-data models for affine nonlinear systems using a fractional-order hold and their zero dynamics

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Abstract: One of the approaches to sampled-data controller design for nonlinear continuous-time systems consists of obtaining an appropriate model and then proceeding to design a controller for the model. Hence, it is important to derive a good approximate sampled-data model because the exact sampled-data model for nonlinear systems is often unavailable to the controller designers. Recently, Yuz and Goodwin have proposed a more accurate model than the simple Euler model in the case of a zero-order hold. This paper derives a sampled-data model for nonlinear systems using a fractional-order hold, and analyzes the zero dynamics of the sampled-data model.
keywords: Nonlinear systems, sampled-data models, zero dynamics, fractional-order hold.

I. INTRODUCTION

Real systems are usually modeled by using laws of physics and are consequently described by ordinary differential equation models. However, we typically interact with these systems by digital devices that utilize sampled data. Thus, the study of sampled-data control systems has become an important issue in control fields.

When dealing with sampled-data models for nonlinear systems, the exact sampled-data model is often unavailable to the controller designers [1]. Then, the accuracy of the approximate sampled-data model has proven to be a key issue in the context of control design, where a controller designed to stabilize an approximate model may fail to stabilize the exact discrete-time model [2].

Recently, Yuz and Goodwin have proposed a more accurate model than the simple Euler model [3]. The resulting model includes extra zero dynamics corresponding to the relative degree of the continuous-time nonlinear system. Such extra zero dynamics are called sampling zero dynamics. It is shown that they are unstable when the relative degree of a continuous-time nonlinear plant is greater than or equal to two. Thus, the closed-loop system becomes unstable when a discrete-time controller design method based on the assumption of the stability of the zero dynamics is applied [5], [6].

In the linear case, the properties of the sampling zeros corresponding to the sampling zero dynamics for nonlinear systems have been discussed in many papers [4], [7]-[13]. Some of the previous studies show that when the relative degree of a continuous-time plant is two, use of a fractional-order hold instead of a zero-order hold overcomes the problem above [8], [9], [13]. Hence it is natural to raise a question how the results of the linear case with the fractional-order hold can be extended to nonlinear systems.

This paper derives a sampled-data model for nonlinear systems using a fractional-order hold, and analyzes the zero dynamics of the sampled-data model to show a condition which assures the stability of the sampling zero dynamics of the model.

II. SAMPLED-DATA MODELS WITH FROH

Consider a class of the following single-input single-output nth-order nonlinear system with the uniform relative degree $r(\leq n)$ which is expressed in its so-called normal form [14], [15]

$$\left\{\begin{array}{rcl} \dot{\boldsymbol{\xi}} & = & \begin{bmatrix} \mathbf{0}_{r-1} & I_{r-1} \\ 0 & \mathbf{0}_{r-1}^T \end{bmatrix} \boldsymbol{\xi} \\ & & + \begin{bmatrix} \mathbf{0}_{r-1} \\ 1 \end{bmatrix} (b(\boldsymbol{\xi},\ \boldsymbol{\eta}) + a(\boldsymbol{\xi},\ \boldsymbol{\eta})u) \\ \dot{\boldsymbol{\eta}} & = & \boldsymbol{c}(\boldsymbol{\xi},\ \boldsymbol{\eta}) \\ y & = & \xi_1 \end{array}\right. \tag{1}$$

$$\boldsymbol{\xi} = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_r \end{bmatrix}, \ \boldsymbol{\eta} = \begin{bmatrix} \eta_{r+1} \\ \vdots \\ \eta_n \end{bmatrix}, \boldsymbol{c} = \begin{bmatrix} c_{r+1}(\boldsymbol{\xi},\ \boldsymbol{\eta}) \\ \vdots \\ c_n(\boldsymbol{\xi},\ \boldsymbol{\eta}) \end{bmatrix}$$

where $a(\boldsymbol{\xi},\ \boldsymbol{\eta}) \neq 0$, $b(\mathbf{0},\mathbf{0}) = 0$ and $\boldsymbol{c}(\mathbf{0},\mathbf{0}) = \mathbf{0}$.

We are interested in the sampled-data model for the nonlinear system (1) when the input is generated by a fractional-order hold (FROH); i.e.,

$$u(t) = u(kT) + \beta \cdot \frac{u(kT) - u((k-1)T)}{T}(t - kT), \quad kT \le t < (k+1)T, \quad k = 0,\ 1, \cdots \quad (2)$$

where β is a real design parameter and T is a sampling period. The signal reconstruction of a fractional-order hold is shown in Fig. 1 [8], [9], [13].

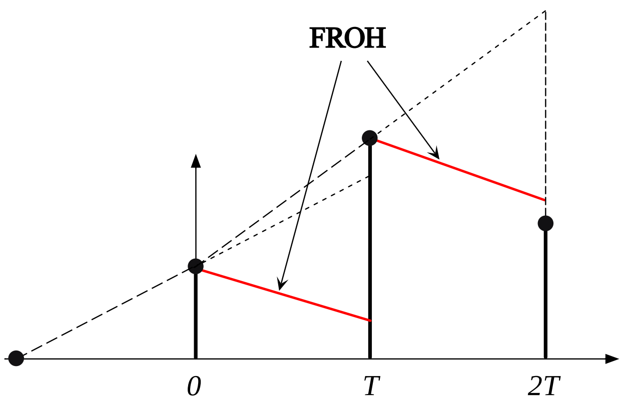

Fig. 1: The signal reconstruction of a fractional-order hold with $\beta = -0.5$.

The sampled-data model for (1) is derived below. First, note that

$$\dot{u}(t) = \beta \cdot \frac{u(kT) - u((k-1)T)}{T}, \ \ddot{u}(t) = 0 \quad (3)$$

then, one can obtain for sufficiently small sampling periods that

$$\begin{cases} \dot{y} &= \dot{\xi}_1 = \xi_2 \\ &\vdots \\ y^{(r-1)} &= \dot{\xi}_{r-1} = \xi_r \\ y^{(r)} &= b + au \\ y^{(r+1)} &= \dot{b} + \dot{a}u + a\dot{u} \\ &= \dot{b} + \dot{a}u + a\beta \cdot \dfrac{u(kT) - u((k-1)T)}{T} \\ &\approx a\beta \cdot \dfrac{u(kT) - u((k-1)T)}{T} \end{cases} \quad (4)$$

The final approximation result of $y^{(r+1)}$ is derived from the fact that the third term with β is dominant in the second equation for a sufficiently small T.

Applying the Taylor's expansion formula to $y^{(i)}((k+1)T)$ and using (4) yield

$$\begin{aligned} \xi_{i+1,k+1} &= y_{k+1}^{(i)} \\ &\approx y_k^{(i)} + Ty_k^{(i+1)} + \frac{T^2}{2}y_k^{(i+2)} + \\ &\quad \cdots + \frac{T^{r-i}}{(r-i)!}y_k^{(r)} + \frac{T^{r-i+1}}{(r-i+1)!}y_k^{(r+1)} \\ &\approx \xi_{i+1,k} + T\xi_{i+2,k} + \frac{T^2}{2}\xi_{i+3,k} + \cdots \\ &\quad + \frac{T^{r-i}}{(r-i)!}(b_k + a_k u_k) \\ &\quad + \frac{T^{r-i}}{(r-i+1)!}\{a_k\beta(u_k - u_{k-1})\} \\ &\quad i = 0, \cdots, r-1 \end{aligned} \quad (5)$$

where

$$b_k \equiv b(\boldsymbol{\xi}_k,\ \boldsymbol{\eta}_k), \ a_k \equiv a(\boldsymbol{\xi}_k,\ \boldsymbol{\eta}_k) \quad (6)$$

and, the subscripts k and $k+1$ denote the time instants kT and $(k+1)T$, respectively.

Hence, a sampled-data model for (1) is obtained as follows.

$$\boldsymbol{\xi}_{k+1} = F_{\beta,k}\boldsymbol{\xi}_k + \boldsymbol{g}_{\beta,k}u_k + \boldsymbol{h}_k \quad (7)$$

$$\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + T\boldsymbol{c}(\boldsymbol{\xi}_k,\ \boldsymbol{\eta}_k) \quad (8)$$

$$y_k = \xi_{1,k} \quad (9)$$

where

$$\boldsymbol{\xi}_k = [\ \xi_{1,k}\ \cdots\ \xi_{r,k}\ u_{k-1}]^T$$

$$F_{\beta,k} = \begin{bmatrix} 1 & T & \frac{T^2}{2!} & \cdots & \frac{T^{r-1}}{(r-1)!} & -\frac{T^r}{(r+1)!}\beta a_k \\ 0 & 1 & T & \cdots & \frac{T^{r-2}}{(r-2)!} & -\frac{T^{r-1}}{r!}\beta a_k \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{T}{2!}\beta a_k \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{g}_{\beta,k} = \begin{bmatrix} \frac{T^r}{r!}a_k + \frac{T^r}{(r+1)!}\beta a_k \\ \frac{T^{r-1}}{(r-1)!}a_k + \frac{T^{r-1}}{r!}\beta a_k \\ \vdots \\ Ta_k + \frac{T}{2!}\beta a_k \\ 1 \end{bmatrix}, \ \boldsymbol{h}_k = \begin{bmatrix} \frac{T^r}{r!}b_k \\ \frac{T^{r-1}}{(r-1)!}b_k \\ \vdots \\ Tb_k \\ 0 \end{bmatrix}$$

The local truncation error between the true system output and the output of the sampled-data model is of order T^{r+1} which implies that the accuracy of the obtained sampled-data model is the same order as that by Yuz and Goodwin's model [3].

III. SAMPLING ZERO DYNAMICS OF THE SAMPLED-DATA MODELS

In this section, we obtain the sampling zero dynamics of the sampled-data model (7)-(9). First, substituting $y_{k+1} = y_k = 0$ into (7) and (8) yields

$$\begin{aligned} \begin{bmatrix} \bar{\boldsymbol{\xi}}_{k+1} \\ 0 \end{bmatrix} &= M \begin{bmatrix} \bar{\boldsymbol{\xi}}_k \\ u_k \end{bmatrix} + \boldsymbol{l} && (10) \\ \boldsymbol{\eta}_{k+1} &= \boldsymbol{\eta}_k + T\boldsymbol{c}(0,\ \bar{\boldsymbol{\xi}}_k,\ \boldsymbol{\eta}_k) && (11) \end{aligned}$$

where

$$\begin{aligned}
\bar{\xi}_k &= \begin{bmatrix} \xi_{2,k} & \cdots & \xi_{r,k} & u_{k-1} \end{bmatrix}^T \\
M &= \begin{bmatrix} M_{11} & \boldsymbol{m}_{12} \\ \boldsymbol{m}_{21}^T & m_{22} \end{bmatrix} \\
M_{11} &= \begin{bmatrix} 1 & T & \frac{T^2}{2!} & \cdots & \frac{T^{r-2}}{(r-2)!} & -\frac{T^{r-1}}{r!}\beta a_{k0} \\ 0 & 1 & T & \cdots & \frac{T^{r-3}}{(r-3)!} & -\frac{T^{r-2}}{(r-1)!}\beta a_{k0} \\ \vdots & & & \ddots & & \vdots \\ 0 & & \cdots & 0 & 1 & -\frac{T}{2!}\beta a_{k0} \\ 0 & & \cdots & & 0 & 0 \end{bmatrix} \\
\boldsymbol{m}_{12} &= \begin{bmatrix} \left(\frac{T^{r-1}}{(r-1)!} + \frac{T^{r-1}}{r!}\beta\right) a_{k0} \\ \left(\frac{T^{r-2}}{(r-2)!} + \frac{T^{r-2}}{(r-1)!}\beta\right) a_{k0} \\ \vdots \\ \left(T + \frac{T}{2!}\beta\right) a_{k0} \\ 1 \end{bmatrix} \\
\boldsymbol{m}_{21} &= \begin{bmatrix} T & \frac{T^2}{2} & \cdots & \frac{T^{r-1}}{(r-1)!} & -\frac{T^r}{(r+1)!}\beta a_{k0} \end{bmatrix}^T \\
m_{22} &= \left(\frac{T^r}{r!} + \frac{T^r}{(r+1)!}\beta\right) a_{k0} \\
\boldsymbol{l} &= \begin{bmatrix} \frac{T^{r-1}}{(r-1)!} b_{k0} & \cdots & T b_{k0} & 0 & \frac{T^r}{r!} b_{k0} \end{bmatrix}^T
\end{aligned}$$

and, a_{k0} and b_{k0} are defined as a_k and b_k, respectively, with $y = 0$. Deleting u_k in (10) leads to the following sampling zero dynamics for sufficiently small sampling periods

$$\bar{\boldsymbol{\xi}}_{k+1} = M_r \bar{\boldsymbol{\xi}}_k \qquad (12)$$

where

$$\begin{aligned}
M_r &= \begin{bmatrix} m_{1,1} & \cdots & m_{1,r} \\ \vdots & \ddots & \vdots \\ m_{r,1} & \cdots & m_{r,r} \end{bmatrix} \\
m_{i,j} &= T^{j-i}\left\{\frac{1}{(j-i)!} - \frac{\gamma_i}{j!}\right\} \\
& \quad i \le j,\ i = 1,\ \cdots,\ r-1,\ j = 1,\ \cdots,\ r-1 \\
m_{i,j} &= -T^{j-i}\frac{\gamma_i}{j!} \\
& \quad i > j,\ i = 1,\ \cdots,\ r-1,\ j = 1,\ \cdots,\ r-1 \\
m_{i,r} &= T^{r-i}\left\{-\frac{1}{(r+1-i)!} + \frac{\gamma_i}{(r+1)!}\right\}\beta a_{k0} \\
& \quad i = 1,\ \cdots,\ r-1 \\
m_{r,j} &= -\frac{1}{T^{r-j}}\frac{\gamma_r}{j! a_{k0}},\ j = 1,\ \cdots,\ r-1 \\
m_{r,r} &= \frac{\beta}{(r+1)+\beta} \\
\gamma_i &= \frac{\frac{(r+1)!}{(r-i)!} + \frac{(r+1)!}{(r+1-i)!}}{(r+1)+\beta},\ i = 1,\ \cdots,\ r-1 \\
\gamma_r &= \frac{(r+1)!}{(r+1)+\beta}
\end{aligned}$$

The matrices M_r are listed below for a few values of r.

$$M_1 = \frac{\beta}{2+\beta} \qquad (13)$$

$$M_2 = \begin{bmatrix} \frac{-3-2\beta}{3+\beta} & -\frac{T}{6+2\beta}a_{k0}\beta \\ -\frac{1}{T}\frac{6}{(3+\beta)a_{k0}} & \frac{\beta}{3+\beta} \end{bmatrix} \qquad (14)$$

$$M_3 = \begin{bmatrix} -\frac{8+3\beta}{4+\beta} & -T\frac{2+\beta}{4+\beta} & \frac{T^2\beta a_{k0}}{3!(4+\beta)} \\ -\frac{1}{T}\frac{24+12\beta}{(4+\beta)} & -\frac{8+5\beta}{4+\beta} & \frac{T\beta a_{k0}}{2!(4+\beta)} \\ -\frac{4!}{T^2(4+\beta)a_{k0}} & -\frac{4!}{T(8+2\beta)a_{k0}} & \frac{\beta}{4+\beta} \end{bmatrix} \qquad (15)$$

As a result, the sampling zero dynamics of the sampled-data model with FROH have the following properties.

Properties 1: Consider the sampled-data model with FROH for a continuous-time system (1) with relative degree r.

Case (i) $r = 1$. The sampling zero dynamics are stable if

$$-1 < \beta \tag{16}$$

Case (ii) $r = 2$. Assume that the term $a(0,\ \dot{y},\ \cdots,\ y^{(r-1)},\ \boldsymbol{\eta})$ is constant. The sampling zero dynamics are stable if

$$-1 < \beta < 0 \tag{17}$$

Case (iii) $r = 3$. The sampling zero dynamics are unstable.

Remark 1: It is easy to see that Properities 1 are similar to the results of the linear case [13].

Remark 2: On the controlled Van der Pol system with relative degree two, the closed-loop system becomes unstable when Yuz and Goodwin's model is used for design of feedback controller which requires the stability of the zero dynamics [6]. However, from the Properties 1 (ii), on such a case, the resulting feedback control system is stable when FROH with $-1 < \beta < 0$ is used for a hold.

IV. CONCLUSION

This paper derives a sampled-data model and analyzes its zero dynamics for nonlinear systems in the case of a fractional-order hold. The resulting model has an advantage to the Yuz and Goodwin's model with respect to the stability of the sampling zero dynamics when the relative degree of a continuous-time nonlinear plant is two.

References

[1] D. S. Laila, D. Nešić and A. Astolfi: Sampled-data control of nonlinear systems, *Advanced Topics in Control Systems Theory, LNCIS 328*, A. Loría *et al.* (Eds.), Springer-Verlag London Limited, pp.91–137, 2006.

[2] D. Nešić and A. Teel: A framework for stabilization of nonlinear sampled-data systems based on their approximate discrete-time models, *IEEE Trans. on Automatic Control*, Vo.49, No.7, pp.1103–1122, 2004.

[3] J. I. Yuz and G. C. Goodwin: On sampled-data models for nonlinear systems, *IEEE Trans. on Automatic Control*, Vo.50, No.10, pp.1477–1489, 2005.

[4] K. J. Åström, P. Hagander and J. Sternby: Zeros of sampled systems, *Automatica*, Vol. 20, No. 1, pp. 31-38, 1983.

[5] M. Ishitobi, M. Nishi and S. Kunimatsu: Stability of zero dynamics of sampled-data nonlinear systems, *Proc. of the 17th IFAC World Congress*, pp.5969–5973, 2008.

[6] M. Nishi, S. Kunimatsu, M. Ishitobi and T. Wada: On sampled-data models for nonlinear systems with relative degree two, *Proc. of the 3rd Int. Symp. on Advanced Control of Industrial Processes*, pp.272–276, 2008.

[7] T. Hagiwara, T. Yuasa and M. Araki: Stability of the limiting zeros of sampled-data systems with zero- and first-order holds, *Int. J. Control*, Vol.58, No.6, pp.1325–1346, 1993.

[8] K. M. Passino and P. J. Antsaklis: Inverse stable sampled low-pass systems, *Int. J. Control*, Vol.47, No.6, pp.1905–1913, 1988.

[9] M. Ishitobi: Stability of zeros of sampled system with fractional order hold, *IEE Proc.-Control Theory and Applications*, Vol.143, No.3, pp.296–300, 1996.

[10] Y. Hayakawa, S. Hosoe and M. Ito: On the limiting zeros of sampled multivariable systems, *Syst. Contr. Lett.*, Vol. 2, No. 5, pp. 292–300, 1983.

[11] S. R. Weller: Limiting zeros of decouplable MIMO systems, *IEEE Trans. on Automatic Control*, Vol. AC-44, No. 1, pp. 129–134, 1999.

[12] M. Ishitobi: A stability condition of zeros of sampled multivariable systems, *IEEE Trans. on Automatic Control*, Vol.45, No.2, pp.295–299, 2000.

[13] M. Ishitobi: Properties of zeros of a discrete-time system with fractional order hold, *Proc. of the 35th IEEE Conference on Decision and Control*, pp.4339–4344, 1996.

[14] A. Isidori: *Nonlinear Control Systems*, 3rd ed. New York: Springer Verlag, 1995.

[15] H. Khalil: *Nonlinear Systems*, 3rd ed., Upper Saddle River, NJ: Prentice-Hall, 2002.