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A Network Visualization of Stable Matching in Stable Marriage Problem

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Abstract: The Stable Marriage Problem (SMP) seeks matching between *n* women and *n* men satisfying a stability, which could otherwise lead to divorce and extramarital affairs. We have introduced a network consisting of nodes representing matching and links between nodes which attains each other by exchanging a partner between two pairs. For visualization, the network is depicted with nodes layouted involving several coordinates such as either women's or men's or both satisfactions. With the network visualization, regularity and symmetry can be made conspicuous in specific instances of SMP such as Latin SMP.

Keywords: network visualization, stable marriage problem, stable matching, Latin square, matching network**.**

I. Introduction

Since the landmark work of Gale and Sharpley [1], stable marriage problem has been studied extensively in several communities such as discrete mathematics, algorithm, operations research, and economics. Not only theoretical work but practical applications of heuristics had been implemented in the real assignment problem of intern to the hospitals without minding the theoretical results [2].

Knuth conjectured that any stable matching may be attained by matching blocking pairs by exchanging partners [3], however, the counterexample had been presented by Tamura [4]. This fact would suggest that the network visualization of the assignment problems including the stable marriage problem might not only clarify the problems themselves but give an insight of the solutions such as symmetries and stabilities embedded in the solution spaces, hence allowing decision making by an authority or by individuals easier.

On the other hand, network visualization demonstrated its power by visualizing large-scale networks such as computer networks, power grids, social networks, etc as well as complex networks such as genetic networks, cell networks, and metabolic networks. Graph drawing algorithms have been studied extensively to draw these large-scale and complex networks (e.g. [5]).

The visualization of network connecting all the nodes corresponding to matching solutions including stable matching would enhance understanding the structure of the solution network such as existence of specific cycles. It would also enhance understanding several solutions such as man-optimal (hence woman-pessimal) and woman-optimal (hence man-pessimal) and other ones between them in the distributive lattice [3] where the partial order between matchings is defined as every woman (man) in a matching is as satisfied with the partner as with the partner in the lower ordered matching. The network visualization may be implemented in several distinct ways depending on the aspect to be visualized using different coordinates measuring man's (woman's) satisfaction and total satisfaction.

The stable marriage problem has been studied by networks ranging from some-what direct expression of bipartite graph to a sophisticated expression of marriage graph [6]. The network of solutions also has been studied by means of the network such as a distributive lattice. We also express the matching by a network with several distinct coordinate-systems. The network visualization of instances of SMP reveals several similarity, regularities and symmetries among instances, which have not been recognized otherwise. One of the notable symmetries is the degeneration of the net-work, where degenerated network is defined by multiple nodes and edges are placed in the common coordinate in the coordinate system.

Section 2 presents stable marriage problems with the definition of stability. Section 3 defines the network whose nodes are matchings and edges partner-exchange. Section 4 explains the coordinates where nodes (matchings) are placed. Section 5 presents several examples of SMP with a regular structure whose matching networks are visualized.

II. Stable Marriage Problem and Stable Matching

The Stable Marriage Problem (SMP) assumes n women and n men each of them has an ordered preference list (or a ranking) without tie to the opposite sex. As in the example shown in Fig. 1, the men m_2 has an ordered preference list (w_3, w_2, w_1, w_4) or a ranking (3, 2, 1, 4), which means m_2 likes w_3 best, and he prefers w_3 to w_2, w_2 to w_1, and w_1 to w_4. One could say that there is an injection (one to one, but not necessarily onto) mapping from a set of women (men) to an element of permutation group of size n such as shown in the ranking by each person (Fig. 1).

Under the above assumptions, SMP seeks for the complete matching between n women and n men (a bijection from n women to n men), which satisfies stability. The stability requires the concept of blocking pair. Two pairs (m_i, w_p) and (m_j, w_q) are blocked by the pair (m_i, w_q) if m_i, prefers w_q to w_p and w_q prefers m_i to m_j, as illustrated in Fig. 2. A complete matching without being blocked is called *stable* matching.

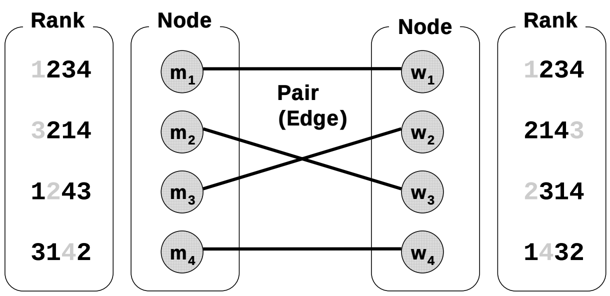

Fig. 1. An illustration of Stable Marriage Problem with size 4.

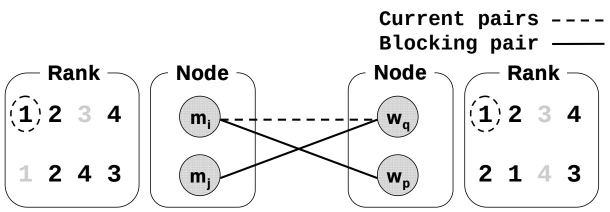

Fig. 2. An illustration of blocking pair.

III. A Network of Matching Solutions

Although instances of men's preference and women's preference can be expressed by networks, we will rather express matching solutions on networks. In the network, each node expresses matching and an edge between two nodes indicates that the matching corresponding the node can be realized by exchanging partners in two pairs of another node linked (Fig. 3). With two sets of n men and n women above, let us consider the following two matchings M_1, M_2:

$$M_1 = \{(m_1,w_1),(m_2,w_3),(m_3,w_2),(m_4,w_4)\}$$
$$M_2 = \{(m_1,w_1),(m_2,w_3),(m_3,w_4),(m_4,w_2)\}$$

The matching M_2 can be attained by exchanging the partners in two pairs: (m_3,w_2), and (m_4,w_4) in the matching M_1, thus two nodes corresponding these two matchings are linked in the network. We will call the network complete when the network includes all possible matchings as nodes and all possible partner-exchange as links.

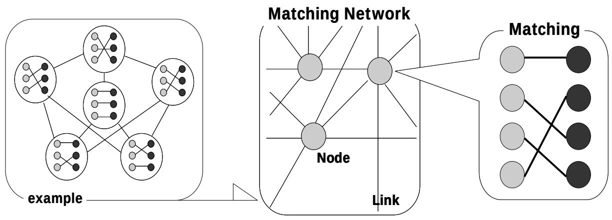

Fig. 3. An example of a matching network with size 3. Bipartite graph indicating each matching is shown inside each node for illustrative purpose.

IV. Coordinates for Network Visualization

Motivation for visualizing matching solutions as a network is to bring regularities and symmetries into daylight of visual perception, which are unseen otherwise. To this end, selection of appropriate coordinates (and its scale) is of great importance. Here, we will use simple and natural coordinates. For example, men's satisfaction P_m is defined as follows:

$$P_m = \sum (n+1-R_{mi})$$

where n is the size of the stable marriage problem and R_{mi} is the man m_i's rank (an integer ranging from 1 to n where 1 means the most favorite) to the current partner in the matching M. Women's satisfaction P_w is similarly defined with R_{wp} the woman w_p's rank to the partner in the matching. These two coordinates may be expressed by one coordinate P_{mw}= P_m－P_w which means men's satisfaction relative to women's satisfaction. Although this P_{mw} is an asymmetric one, the following *E energy* is a symmetric coordinate reflecting both men and women's satisfaction symmetrically.

$$E = \sum (n+1-R_{mi})(n+1-R_{wp})$$

where the summation is taken over all the pair m_i and w_p in the matching M.

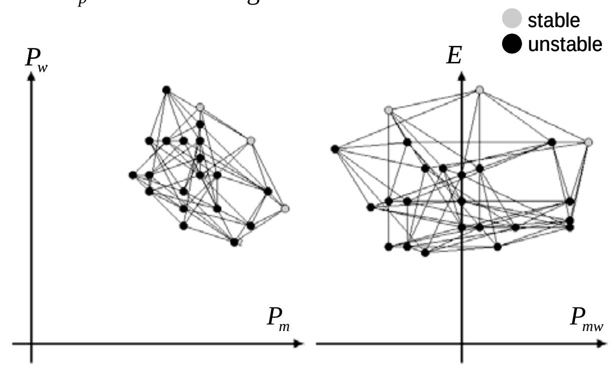

Fig. 4. Network visualization of an instance of SMP with size 4. Left: two coordinates of men's satisfaction P_m and women's satisfaction P_w are used. Right: symmetric coordinate E and asymmetric coordinate P_{mw} are used.

When P_w and P_m are used as two axes (Fig. 4 left), as the points are placed toward more upper-right, the better the solution for either women or men or both. This is similar to indifference curves plotting utility (satisfaction) in microeconomic theory, and to the receiver operating characteristic (ROC) curves in detection theory. Stable solutions are placed on the curve, since they will most often satisfy either women, men or both.

When E and P_{mw} are used as two axes (Fig. 4 right), the points placed higher the better for both women and men, more right the better for men, and more left the better for women. They are upward convex, similarly to the P_w - P_m curve.

It should be noted that the curves include unstable solutions as in Fig. 4, although stable solutions are often included in the curve. Of course, there are exceptions, for it is known that stable matching may not be obtained with any local (local in the sense that individual rank or paired ranks are aggregated) measure such as P_w, P_m, P_{mw} and E. Stable matching nodes are placed at the highest location in E coordinates, and of course there are exceptions too with the same reason.

V. Specific Instances of Stable Marriage Problem

When we visualize matching network of SMP instances with specific structure, some regularities and symmetries will be observed. Latin SMP [7], for example, defines instances where each person's rank to the person in the opposite sex and the rank from the person to the original person add to a constant n+1. Than is,

$$R_{mi,wp} + R_{wp,mi} = n + 1$$

where $R_{mi,wp}$ is the man m_i's rank to the the woman w_p. This equation means a rather strange constraint that the higher a person ranked to persons in the opposite sex, the lower being ranked by them.

Preference structure of SMP may be expressed by a preference matrix { a_{ij} } where the element in i^{-th} row and j^{-th} column a_{ij} is defined to be $R_{mi,wj}$ / $R_{wj,mi}$. For the Latin SMP with size 4, the preference matrix is as follows:

1/4, 2/3, 3/2, 4/1
2/3, 1/4, 4/1, 3/2
3/2, 4/1, 1/4, 2/3
4/1, 3/2, 2/3, 1/4.

For this Latin SMP with size 4, the matching networks are visualized in Fig. 5, where coordinates are drawn in the same manner as in Fig. 4. Regularity and symmetry of the stable matching solutions in the network is made conspicuous. The degeneracy is obvious in P_m - P_w coordinate, for every nodes are stable, and unstable ones are masked by the stable ones. From the definition of the Latin SMP, the preference structure should be symmetric for women and men. This fact is evidently reflected in both figures with P_m - P_w coordinate (Fig. 5 left) and with P_{mw} – E coordinate (Fig. 5 right).

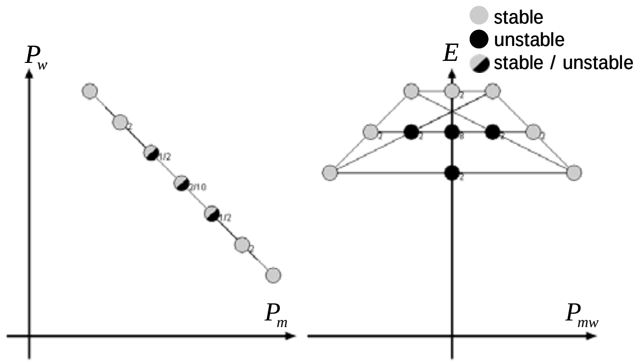

Fig. 5. Network visualization of a Latin SMP with size 4. Left: two coordinates of men's satisfaction P_m and women's satisfaction P_w are used. Stable matchings are shown by squares, while unstable ones by circles in the left to avoid masking when overlapped. Right: symmetric coordinate E and asymmetric coordinate P_{mw} are used.

The followings are three preference matrices of SMP of size 4 with regularities:

1/2,2/3,3/4,4/1 1/3,2/2,3/1,4/4 1/3,2/2,3/1,4/4
2/3,1/2,4/1,3/4 2/4,1/1,4/2,3/3 2/2,1/3,4/4,3/1
3/4,4/1,1/2,2/3 3/1,4/4,1/3,2/2 3/1,4/4,1/3,2/2
4/1,3/4,2/3,1/2 4/2,3/3,2/4,1/1 4/4,3/1,2/2,1/3.

These three preference matrices are obtained by fixing the men's rank to women but shifting and rotating the women's rank to men in several ways in the Latin SMP above. These three SMPs with the above preference matrices (left, middle, right) are visualized in Fig. 6 (above, middle, below, respectively).

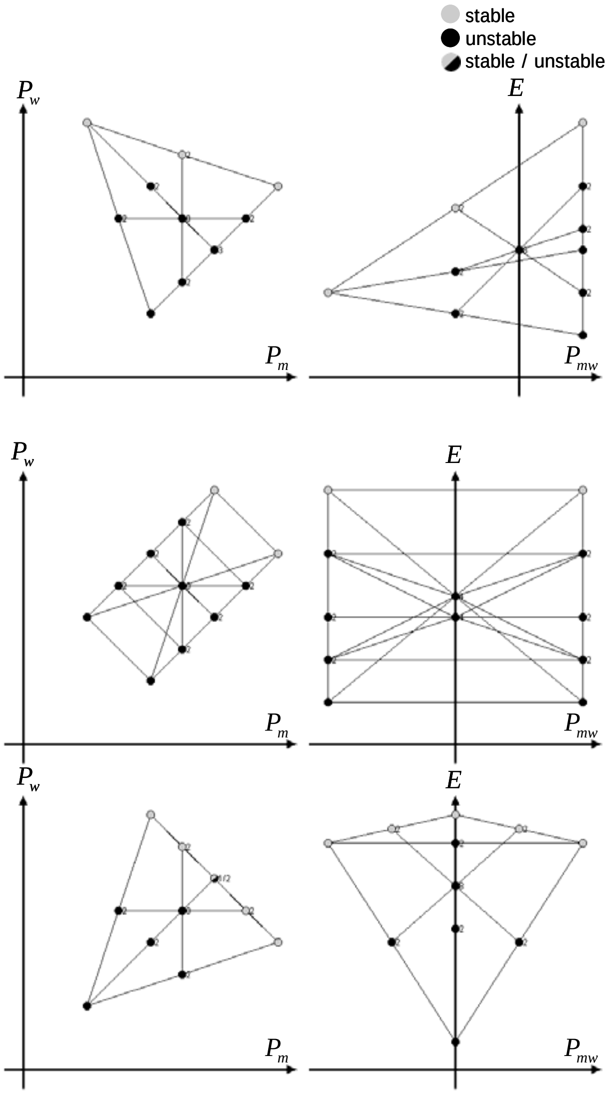

Fig. 6. Network visualization of three instances of SMP with regular structures. Left: two coordinates of men's satisfaction P_m and women's satisfaction P_w are used. Right: symmetric coordinate E and asymmetric coordinate P_{mw} are used.

It can be observed that the first SMP (Fig. 6 above) and the last SMP (Fig. 6 below) share the similar preference structure such as isomorphic with 90 degree anti-clock wise rotation in P_m - P_w coordinate, however, the last SMP is symmetric for men and women while the first one is not as observed with P_{mw} – E coordinate. The symmetry for women and men can be observed for the middle SMP, too. With the preference matrices alone, these facts may not be as obvious as seen in the network visualization.

VI. Summary

The preference structures are involved in the two matrices: preference from women to men; and that from men to women. Although the preference structure itself can directly be expressed by a weighted bipartite graph or stable marriage graph, the matchings are expressed by the matching network whose nodes are matchings and edges partner-exchange. By visualizing the network with adequate nodes layout in the coordinate, the space composed by matching (including stable ones) will be understood geometrically.

As we have glimpsed, symmetry in the preference structure will be understood from the viewpoint of symmetry in the graph visualizing the matching network with an adequate coordinate for the nodes layout.

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