# Fuzzy Sliding Mode Control for Under-Actuated System with Mismatched Uncertainties

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***Abstract***: This paper presents a robust fuzzy sliding mode controller. The methodology of sliding mode control provides an easy way to control under-actuated nonlinear systems with uncertainties. The structure of sliding surface is designed as follows. First, decouple the entire system into second-order systems such that each subsystem has a separate control target expressed in terms of a sliding surface. Second, from sliding surface of subsystems, organize the main sliding surface system. Third, generates a control input for main sliding surface to make whole subsystems moving toward their sliding surface. A fuzzy controller is used for obtaining smooth boundary layer of sliding surface. Finally presented fuzzy sliding mode controller is used to control under-actuated nonlinear system and confirms the validity of the proposed approach and its robustness for uncertainties.

***Keyword***: Robust controller, Sliding mode controller, Fuzzy controller

## I. INTRODUCTION

There are needs for machines which have missions in extremely harsh situations that human can never stand. A robust controller is studied to withstand those kinds of situations. The sliding mode control is a powerful and robust nonlinear feedback control method. It has been developed and applied to feedback control systems for last three decades. The sliding mode controller is insensitive to internal parameter variation and external disturbances when system is on sliding surface. [1], [2]

Mechanical systems with fewer number of control inputs than the number of degree of freedom which should be controlled are called under actuated systems. These systems can be found in industry areas easily. However, their unexpected properties such as nonlinearity and coupling effect make design procedure difficult. In this paper inverted double pendulum will be considered as an example of a class of under-actuated systems. [1], [3]

In this paper, the under-actuated system is controlled using decoupled sliding mode controller. With proposed controller, the system can be controlled simple fuzzy controller which has an equivalent expression of a classical sliding mode control with a boundary layer. It is possible to remove high frequency switching (chattering phenomena) by introducing our fuzzy controller.

This paper is organized as follows. In section II, the design of decoupled controller motivated by [1] is presented. Section III shows design of fuzzy controller is equivalent to boundary layer sliding mode controller [4]. In section IV, the proposed controller is used to control double inverted pendulums. Conclusions are in the last section.

## II. DECOUPLED SLIDING MODE CONTROL

Consider an under-actuated nonlinear system as follows:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t) \\
\dot{x}_2(t) &= f_1(\mathbf{x}) + b_1(\mathbf{x})u(t) + d_1(t) \\
\dot{x}_3(t) &= x_4(t) \\
\dot{x}_4(t) &= f_2(\mathbf{x}) + b_2(\mathbf{x})u(t) + d_2(t) \\
\dot{x}_5(t) &= x_6(t) \\
\dot{x}_6(t) &= f_3(\mathbf{x}) + b_3(\mathbf{x})u(t) + d_3(t)
\end{aligned} \tag{1}$$

Where $\mathbf{x} = [x_1(t), x_2(t), x_3(t), x_4(t)]$ are the state variables; $f_1(\mathbf{x}), f_2(\mathbf{x}), b_1(\mathbf{x})$, and $b_2(\mathbf{x})$ are bounded nominal nonlinear functions; $u(t)$ is the control input; and $d_1(t)$ and $d_2(t)$ are the lumped disturbances, which include the system uncertainties and external disturbances; and assume they are bounded by some positive constants $\varepsilon_i$ $(i = 1,\ 2)$ This system can be considered as two subsystems with second-order canonical form.

Define a pair of sliding surfaces as

$$s_1 = c_1 x_1 + x_2 \tag{2}$$
$$s_2 = c_2 x_3 + x_4 \tag{3}$$
$$s_3 = c_3 x_5 + x_6 \tag{4}$$

where $c_1, c_2, c_3$ are positive constants. From (1) time derivatives of (2), (3) and (4) are obtained as

$$\dot{s}_1 = c_1\dot{x}_1 + \dot{x}_2 = c_1 x_2 + f_1(\mathbf{x}) + b_1(\mathbf{x})u(t) + d_1(t) \tag{5}$$
$$\dot{s}_2 = c_2\dot{x}_3 + \dot{x}_4 = c_2 x_4 + f_2(\mathbf{x}) + b_2(\mathbf{x})u(t) + d_2(t) \tag{6}$$

$$\dot{s}_3 = c_3\dot{x}_5 + \dot{x}_6 = c_3 x_6 + f_3(\mathbf{x}) + b_3(\mathbf{x})u(t) + d_3(t). \quad (7)$$

Assuming the lumped disturbance in (4) is ignored, i.e., $d_1(t) = 0$ by the condition of sliding mode $\dot{s}_1 = 0$, an equivalent control law can be obtained as

$$u_{eq} = \frac{-c_1 x_2 - f_1(\mathbf{x})}{b_1} \quad (8)$$

However, this control only considers the control of subsystem defined by $(x_1, x_2)$ while disregard lumped disturbance and control of $(x_3, x_4)$ and $(x_5, x_6)$ subsystems.

To solve problem stated above, define a hierarchical coupled sliding surface and its derivative as

$$S = s_1 - \eta_2 s_2 - \eta_2 s_3 \quad (9)$$

$$\begin{aligned} \dot{S} &= \dot{s}_1 - \eta_2\dot{s}_2 - \eta_3\dot{s}_3 \\ &= c_1 x_2 + f_1(\mathrm{x}) + b_1(\mathrm{x})u(t) + d(t) - \eta_2\dot{\bar{s}}_2 - \eta_3\dot{\bar{s}}_3 \end{aligned} \quad (10)$$

where $\eta_2$ and $\eta_3$ are positive constants and referred to as coupling factor, $d(t)$ is total uncertainty in $\dot{S}$ i.e., $d(t) = d_1(t) + d_2(t) + d_3(t)$ and $\dot{\bar{s}}_i$ $(i = 2,3)$ are same as (6) and (7) with removing each disturbance $d_2(t)$, $d_3(t)$. These $\eta_2$ and $\eta_3$ are important parameters which will play a main role for the interaction among sliding surfaces $s_1$, $s_2$, and $s_3$ which are shown in Fig. 1.

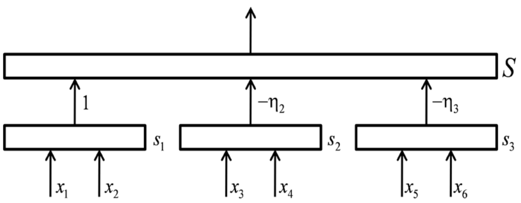


Fig.1. Structure of hierarchical sliding surfaces

The total control input is defined as

$$u = u_{eq} + u_{sw} \quad (11)$$

where $u_{eq}$ is given in (6), and $u_{sw}$ is switching control term to deal with the lumped disturbance and interactive coupling influence.

Consider a Lyapunov function

$$V = \frac{1}{2}S^2. \quad (12)$$

From Lyapunov theorem if $\dot{V}$ is negative definite, the system trajectory will be driven and attracted toward the sliding surface and remain sliding on it until the origin is reached asymptotically.

From (9), (10), and (11) the Lyapunov stability condition can be derived as follows:

$$\begin{aligned} \dot{V} &= S\dot{S} \\ &= S[c_1 x_2 + f_1 + b_1 u + d_1 - \eta_2\dot{s}_2 - \eta_3\dot{s}_3] \\ &= S[c_1 x_2 + f_1 + b_1(u_{eq} + u_{sw}) + d - \eta_2\dot{s}_2 - \eta_3\dot{s}_3] \\ &= S[b_1 u_{sw} + d - \eta_2\dot{s}_2 - \eta_3\dot{s}_3] \end{aligned} \quad (13)$$

Choose $u_{sw}$ as follows:

$$u_{sw} = \frac{-\varepsilon\,\mathrm{sgn}(S) - \eta_2\dot{s}_2 - \eta_3\dot{s}_3}{b_1} \quad (14)$$

where sgn(•) is a sign function and $\varepsilon$ is positive number whose value is larger than supremum of total uncertainty $d$.

Then (13) becomes

$$\begin{aligned} \dot{V} &= -\varepsilon\,\mathrm{sgn}(S)S + d_1 S \\ &\le -(\varepsilon - d_{1\max})|S| \\ &\le -\delta \end{aligned} \quad (15)$$

where $\delta$ is positive number.

In summary, the control law is given as

$$u = \frac{-c_1 x_2 - f_1(\mathrm{x}) - \eta_2\dot{s}_2 - \eta_3\dot{s}_3 - \varepsilon\,\mathrm{sgn}(S)}{b_1} \quad (16)$$

## III. FUZZY SLIDING MODE CONTROL

In this section, we propose the fuzzy controller for sliding surface boundary layer. It can be shown a particular fuzzy controller is an extension of an SMC with a boundary layer [4]. Through decoupling sliding mode control, it can be easily implemented using simple fuzzy rules.

Suppose the fuzzy controller is constructed from the following IF-THEN rules.

$R^1$: if $s$ is NB, then $u_f$ is BIGGER

$R^2$: if $s$ is NM, then $u_f$ is BIG

$R^3$: if $s$ is ZR, then $u_f$ is MEDIUM

$R^4$: if $s$ is PM, then $u_f$ is SMALL

$R^5$: if $s$ is PB, then $u_f$ is SMALLER

where NB is negative big, NM is negative medium, ZR is zero, PM is positive medium, PB is positive big.

Let X and Y be the input and output space of fuzzy rules. For any arbitrary fuzzy set $\tilde{F}$ in X, each rule $R^i$ can determine a fuzzy set $\tilde{F}_x \circ R^i$ in Y. Use sup-min compositional rule of inference with condition that $\tilde{F}_x$

is fuzzy singleton. Then

$$\begin{aligned} \mu_{\tilde{F}_x \circ R^i}(\mu_f) &= \min[\mu_{\tilde{F}_s^i}(\alpha), \mu_{\tilde{F}_{\mu_f}^i}(\alpha)] \\ \mu_{\tilde{F}_{\mu_f}^d}(\mu_f) &= \max[\mu_{\tilde{F}_x \circ R^1}(\mu_f), \ldots.., \mu_{\tilde{F}_x \circ R^5}(\mu_f)] \end{aligned} \tag{17}$$

The crisp output is obtained by center-of-area defuzzifier

$$u_c = \frac{\int u_f \cdot \mu_{\tilde{F}_{u_f}^d}(u_f) du_f}{\int \mu_{\tilde{F}_{u_f}^d}(u_f) du_f} \tag{18}$$

Then $u_c$ has the following form:

$$u_c = \tilde{u} - K_f f_{fuzzy}(s) \tag{19}$$

where

$$f_{fuzzy}(x) = \begin{cases} -1 & \text{if } x < -1 \\ -\frac{1}{2}\frac{(2x+3)(3x+1)}{4x^2+6x+1} & \text{if } -1 \le x < -1/2 \\ -\frac{1}{2}\frac{x(2x+3)}{4x^2+2x-1} & \text{if } -1 \le x < -1/2 \\ \frac{1}{2}\frac{x(2x-3)}{4x^2-2x-1} & \text{if } 0 \le x < 1/2 \\ \frac{1}{2}\frac{(2x-3)(3x-1)}{4x^2-6x+1} & \text{if } 1/2 \le x < 1 \\ 1 & \text{if } x \ \ge 1 \end{cases}$$

This controller has a nearly same property of boundary layer sliding mode controller with boundary width 1 as shown in Fig. 2

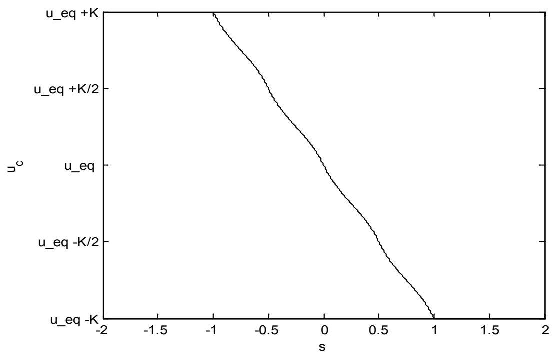


Fig.2 Result of fuzzy controller

The controller shown in (16) will have high frequency switching (chattering phenomena) near the sliding surface due to $\mathrm{sgn}(\bullet)$ function. These rapid changes of input can be avoided by introducing boundary layer method. The fuzzy controller (19) has same form of sliding mode controller (16). By replacing sign function $\mathrm{sgn}(\bullet)$ with $f_{fuzzy}(s)$, we can easily obtain a boundary layer and we can remove chattering phenomena.

## IV. SIMULATION RESULT

In this section, we shall show that proposed method is applicable to double inverted pendulums system which is typical example of a nonlinear under-actuated system. The structure of the system is illustrated in Fig. 3.

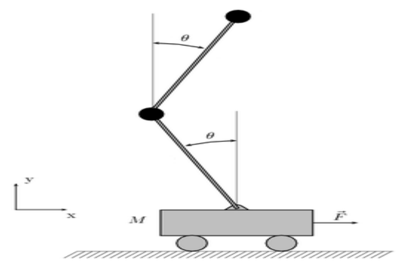

Fig. 3 Structure of double inverted pendulums

The system's dynamics can be represented as

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= f_1 + b_1 u + d_1 \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= f_2 + b_2 u + d_2 \\ \dot{x}_5 &= x_6 \\ \dot{x}_6 &= f_3 + b_3 u + d_3 \end{aligned} \tag{20}$$

where $x_1 = \theta_1$ is angle of lower pole with respect to the vertical axis, $x_2 = \dot{\theta}_1$ is angular velocity of lower pole with respect to the vertical axis; $x_3 = \theta_2$ is angle of upper pole with respect to the vertical axis, $x_4 = \dot{\theta}_2$ is angular velocity of upper pole with respect to the vertical axis; $x_5 = x$ is the position of cart, $x_6 = \dot{x}$ is velocity of cart; $u$ is control input, and $f_i$, $b_i$ is given in [3] and $d_i$ is the mismatched uncertain term whose bound is known.

For simulation, the system parameter are chosen as the cart mass $M = 1kg$, lower pole mass $m_1 = 1kg$, upper pole mass $m_2 = 1kg$, the lower pole length $l_1 = 1m$, the upper pole length $l_2 = 1m$ the gravitational acceleration $g = 9.81 m \cdot s^{-2}$ and $d_i = 0.0872\rho$ where $\rho$ is a random number whose range is $(-1,1)$ and the total mismatched uncertainty $d$ is bounded by 0.3, i.e., $|d| \le 0.3$.

The parameters for presented fuzzy sliding mode controller are selected as sliding surface parameter $c_1 = 6$, $c_2 = 0.4$, $c_3 = 0.15$ and interaction parameter $\eta_2 = 0.016$, $\eta_3 = 0.676$, $\varepsilon = 5$ after trial and error. The initial values are:

$\theta_1 = \pi/6$, $\dot{\theta}_1 = 0$, $\theta_2 = \pi/18$, $\dot{\theta}_2 = \pi/18$, $x = 0$, $\dot{x} = 0$.

Our control objective is making the system from the initial state $(\pi/6, 0, \pi/18, 0, 0, 0)$ to the desired states $(0,0,0,0,0,0)$. Fig. 4 and Fig. 5 show the simulation result.

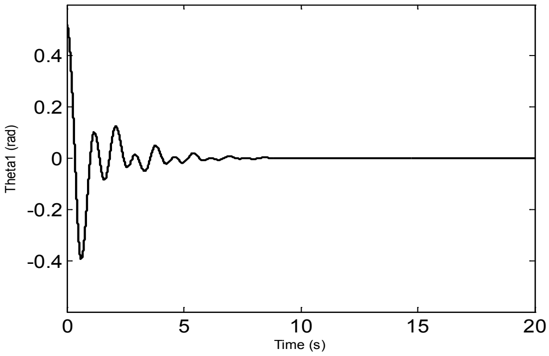


Fig. 4(a) Angle of lower pole

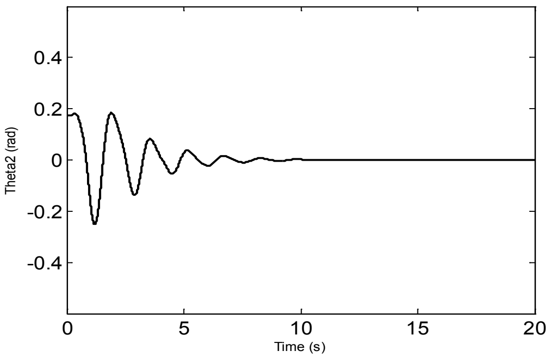


Fig. 4(b) Angle of upper pole

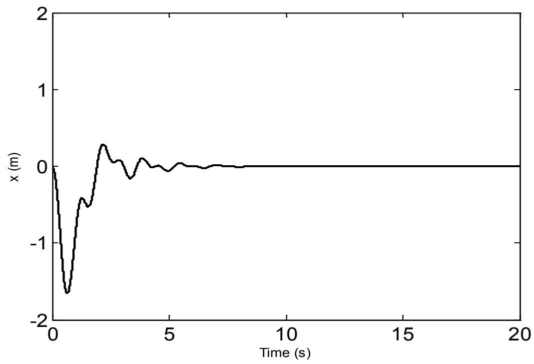


Fig. 4(c) Cart position

Fig. 4(a) and Fig. 4(b) show the proposed controller can make the double inverted pendulums upright. Fig. 4(c) shows the cart position is well controlled as well.

Fig. 5 shows control effort for moving cart. As the graph shows, the presented controller alleviate chattering problem effectively. Furthermore, the figure shows that presented method use relatively small control force.

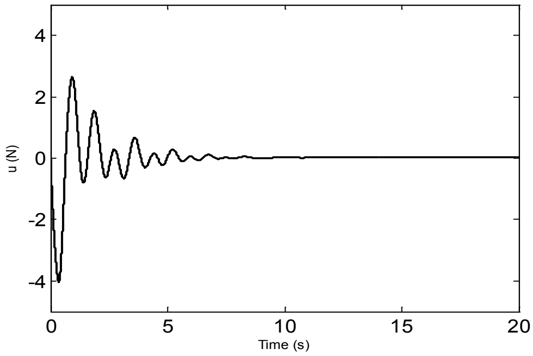


Fig. 5 control force

## V. CONCLUSION

Fuzzy sliding mode controller has been proposed for a class of under-actuated systems. By introducing hierarchical switching scheme, it is possible to decouple the whole system into subsystems which has canonical form. The advantage of this approach is that the control law is determined by lyapunov function, so this approach can guarantee the system stability. By decoupling sliding surface, fuzzy rule based controller design can be easily obtained. A fuzzy controller is used for obtaining smooth boundary of sliding surface. It is shown that chattering problem can be solved using fuzzy scheme. Proposed method is well applied to double inverted pendulum which is typical example of highly nonlinear under-actuated system.

## VII. ACKNOWLEDGMENT

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