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Robust Control Method for the Inverted Pendulum System with Structured Uncertainty Caused by Measurement Error

N. Takahashi, T. Morikubo, O. Sato and M. Kono

Department of Computer Science and Systems Engineering,
Faculty of Engineering, University of Miyazaki,
1-1 Gakuenkibanadai Nishi, Miyazaki, 889-2192, Japan
E-mail : ntaka@cs.miyazaki-u.ac.jp

Abstract

In this paper, we propose a design method of the inverted pendulum system with structured uncertainty. We consider such uncertainty is caused by a measurement error in the rotation angle of the pendulum and causes to the system structure that can not be included in the nominal parameter. For the obtained uncertain system, we apply the integral tracking control and guaranteed cost control to design a robust stable tracking control system. At the last, we show the effectiveness of our method through numerical example.

1 Introduction

In the robust control problem, it is important how to evaluate effect of uncertainty. Such uncertainty effects a bad influence to the system performance. In general, such uncertainty is not contained in the nominal system, then, for the design of the robust stable control system, it is need to express the structural property of the instrument. Kimura showed the derivation method of the structured uncertainty that is caused from the higher order terms in the Taylor expansion[1]. In this paper, we consider the effect of uncertainty to the performance of system that is caused by measurement error. The effect of uncertainty is formulated as the structured uncertainty corresponds to the system matrices. For this system, to show the effectiveness of our proposed method, we apply the integral tracking control system and guaranteed cost control design method.

In section 2, we will propose the formulate method of the robust inverted pendulum-car system model which included uncertain element. In section 3, we will apply the integral tracking control system to the obtained system, and guaranteed cost control method. In section 4, we will present a numerical solution and a simulation result.

2 Derivation of the Uncertain Inverted Pendulum Model

In this section, we will derivate the linear system of the inverted pendulum-car model with structured uncertainties. Now we consider the inverted pendulum-car model that takes following parameters,

M : a Mass of the Car [kg]

m : a Mass of the Pendulum [kg]

I : the Inertia Moment of the Pendulum [$\mathrm{kg \cdot m^2}$]

L : the Distance from the Rotation Axis to the Center of Gravity of Pendulum [m]

g : Gravity [$\mathrm{m/sec^2}$]

$\theta(t)$: an Angle of the Pendulum [rad]

$u(t)$: an External Force on the Car [N] (Input)

$z(t)$: a Position of the Car [m] (Output)

The system state variables are obtained by the measured value of sensors in the instrument or observation equipments, e.g. potentiometer, image data of video camera. Unfortunately, these measurement values contain measurement deviation caused by a lower resolution limit of the sensor, noise of sensor and so on. Then it needs to include the effects of such disturbance in the system dynamics. To deal with this problem, we will introduce following uncertainty in the angle of pendulum.

$$\theta(t) = \theta_0(t) + \varDelta\theta \tag{1}$$

where, $\theta(t)$ is a measured angle of the pendulum consist with $\theta_0(t)$ and $\varDelta\theta$. $\theta_0(t)$ is a nominal element and

$\Delta\theta$ is a disturbance element which caused by measurement error. From the fundamental formulae of the trigonometric functions, we have

$$\begin{aligned}\sin(\theta(t)) &= \sin(\theta_0(t)+\Delta\theta)\\ &= \sin(\theta_0(t))\cos(\Delta\theta)+\cos(\theta_0(t))\sin(\Delta\theta),\\ \cos(\theta(t)) &= \cos(\theta_0(t)+\Delta\theta)\\ &= \cos(\theta_0(t))\cos(\Delta\theta)-\sin(\theta_0(t))\sin(\Delta\theta)\end{aligned}$$

Here we assume that $\Delta\theta$ takes very small value, then $\sin(\Delta\theta)\to 0$. We estimate a maximum variation value of $\cos(\Delta\theta)$ as constant value Δc. From the linearization by using Taylor expansion, we have $\sin(\theta_0(t)) \approx \theta_0(t), \cos(\theta_0(t)) \approx 1$. Thus,

$$\sin(\theta(t)) \approx \Delta c\cdot\theta_0(t), \tag{2}$$

$$\cos(\theta(t)) \approx \Delta c, \tag{3}$$

$$\frac{d^2}{dt^2}(\sin(\theta(t))) \approx \Delta c\cdot\ddot{\theta}_0(t), \tag{4}$$

$$\frac{d^2}{dt^2}(\cos(\theta(t))) \approx 0. \tag{5}$$

A dynamics of the inverted pendulum-car system is expressed as

$$\begin{aligned}I\ddot{\theta}_0(t) &= VL\sin(\theta_0(t))-HL\cos(\theta_0(t)),\\ H &= m\ddot{z}(t)+ML\cdot\frac{d^2}{dt^2}(\sin(\theta_0(t))),\\ V &= mL\cdot\frac{d^2}{dt^2}(\cos(\theta_0(t)))+mg,\\ H &= u-M\ddot{z}(t).\end{aligned}$$

In virtue of (2)-(5), we have

$$\begin{aligned}\ddot{z}(t) &= -\frac{\Delta c^2\cdot m^2gL^2}{I(m+M)+\Delta c^2\cdot mML}\theta_0(t)\\ &\quad +\frac{\Delta c^2\cdot mL^2+I}{I(m+M)+\Delta c^2\cdot mML}u(t),\end{aligned} \tag{6}$$

$$\begin{aligned}\ddot{\theta}_0(t) &= \frac{\Delta c\cdot mgL(M+m)}{I(m+M)+\Delta c^2\cdot mL^2}\theta_0(t)\\ &\quad -\frac{\Delta c\cdot mL}{I(m+M)+\Delta c^2\cdot mL^2}u(t).\end{aligned} \tag{7}$$

Here we define the state vector of a linear system is

$$\boldsymbol{x}(t) = [\ \theta_0(t)\ \ \dot{\theta}_0(t)\ \ z(t)\ \ \dot{z}(t)\ ]^{\mathrm{T}}$$

and the system matrices are

$$A(\xi) = \begin{bmatrix} 0 & 1 & 0 & 0\\ \frac{\Delta c\cdot mgL(M+m)}{I(m+M)+\Delta c^2\cdot mL^2} & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ -\frac{\Delta c^2\cdot m^2gL^2}{I(m+M)+\Delta c^2\cdot mL^2} & 0 & 0 & 0\end{bmatrix},$$

$$B(\zeta) = \begin{bmatrix} 0\\ -\frac{\Delta c\cdot mL}{I(m+M)+\Delta c^2\cdot mL^2}\\ 0\\ \frac{\Delta c^2\cdot mL^2+I}{I(m+M)+\Delta c^2\cdot mL^2}\end{bmatrix}.$$

Here we clarified a structure of the effect of the uncertainty about axis of the pendulum. Note that if $\Delta\theta = 0$, that is $\Delta c = 1$, $A(\xi)$ and $B(\zeta)$ equivalent to nominal system A_0 and B_0. Let A_D and B_D are disturbanced system with the maximum value of the variety of uncertainty $\Delta\theta_0$,

$$\begin{aligned}A_1 &= A_0 - A_D,\\ B_1 &= B_0 - B_D.\end{aligned}$$

Consequently, we obtain LTI system with structured uncertainties.

$$\dot{\boldsymbol{x}}(t) = A(\xi)\boldsymbol{x}(t) + B(\zeta)\boldsymbol{u}(t) \tag{8}$$

where, $A(\xi)$ and $B(\zeta)$ are real matrices of appropriate size.

$$A(\xi) = A_0 + \Delta A, \tag{9}$$

$$B(\zeta) = B_0 + \Delta B. \tag{10}$$

A_0 and B_0 are the nominal elements, ΔA and ΔB are the uncertain elements of the system.

$$\Delta A = \sum_{i=1}^{p}\xi_i A_i,\ |\xi_i| \le 1,\ p = 1, \tag{11}$$

$$\Delta B = \sum_{j=1}^{q}\zeta_j B_j,\ |\zeta_j| \le 1,\ q = 1. \tag{12}$$

ξ_i and ζ_j are scalar values which denote the size of uncertainties. A_i and B_j are matrices of structure of the uncertainties.

Here we define output $y(t) = z(t)$, then the output matrix C is obtained as

$$y(t) = C\boldsymbol{x}(t) = [\ 0\ \ 0\ \ 1\ \ 0\ ]\,\boldsymbol{x}(t) \tag{13}$$

All system matrices are of appropriate dimension.

3 Stabilization of the Inverted Pendulum System

In this section, for the structured uncertain inverted pendulum system, we synthesize the integral tracking control system. Next, we design the robust control system by using guaranteed cost control methd[2].

3.1 The Integral Tracking Control System

In this paper, we assume that the controller can use the system state vector $\boldsymbol{x}(t)$. Input signal to the system is $v(t) = v$. Error $e(t)$ of output signal of nominal system $y(t)$ and
input signal $v(t)$ is given as

$$e(t) = v - y(t) = v - C\boldsymbol{x}(t).$$

The derivative of the deflection $e(t)$ is

$$\dot{e}(t) = -C\dot{\boldsymbol{x}}(t) \tag{14}$$

Here we differentiate the system (8), we have

$$\ddot{\boldsymbol{x}}(t) = A(\xi)\dot{\boldsymbol{x}}(t) + B(\zeta)\dot{\boldsymbol{u}}(t) \tag{15}$$

From (14) and (15), we obtain the augmented system

$$\dot{\boldsymbol{x}}_e(t) = A_e(\xi)\boldsymbol{x}_e(t) + B_e(\zeta)u_e(t) \tag{16}$$

where input vector is $u_e(t) = \dot{u}(t)$, state vector is

$$\boldsymbol{x}_e(t) = [\ \dot{\boldsymbol{x}}(t)^{\mathrm{T}} \quad e(t)\ ]^{\mathrm{T}}$$

System matrices are defined as follows

$$\begin{aligned} A_e(\xi) &= \begin{bmatrix} A(\xi) & \mathrm{O}_{4\times1} \\ -C & 0 \end{bmatrix}, \\ B_e(\zeta) &= \begin{bmatrix} B(\zeta) \\ 0 \end{bmatrix}, \end{aligned}$$

Augmented nominal and uncertain system matrices A_{e0}, B_{e0}, A_{ei} and B_{ej} are defined as same form in (11) and (12) corresponding to based system matrices A_0, B_0, A_i and B_j.

3.2 Stabilization by using Guaranteed Cost Control

Here, we apply the guaranteed cost control design method to the augmented integral tracking control system (16). Let us consider the following performance index.

$$J = \int_0^\infty \{\boldsymbol{x}_e^{\mathrm{T}}(t)Q\boldsymbol{x}_e(t) + u_e^{\mathrm{T}}(t)Ru_e(t)\}dt \tag{17}$$

The stochastic algebraic Riccati equation based on the eigenvalue upperbound is

$$\begin{aligned} C^{\mathrm{T}}C + PA_{e0} + A_{e0}^{\mathrm{T}}P - PB_{e0}R^{-1}B_{e0}^{\mathrm{T}}P \\ + U_E(\mathit{\Delta}A_e(\xi), \mathit{\Delta}B_e(\zeta), P, R) = \mathrm{O}_{5\times5} \end{aligned} \tag{18}$$

where, $\mathit{\Delta}A_e(\xi)$ and $\mathit{\Delta}B_e(\zeta)$ are uncertainties defined in (11) and (12). Upper bound matrix $U_E(\cdot)$ is

$$\begin{aligned} U_E(\mathit{\Delta}A_e(\xi), \mathit{\Delta}B_e(\zeta), P, R) \\ = \sum_{i=1}^{p} L_i|\Lambda_i|L_i^{\mathrm{T}} + \sum_{j=1}^{q} M_i|\Gamma_i|M_i^{\mathrm{T}} \end{aligned} \tag{19}$$

where, $|\cdot|$ denotes matrix that have absolute value of each elements. L_i, M_i, Λ_i and Γ_i are

$$L_i^{\mathrm{T}}(PA_{ei} + A_{ei}^{\mathrm{T}}P)L_i = \Lambda_i \tag{20}$$

$$M_i^{\mathrm{T}}P(B_{ei}R^{-1}B_{e0}^{\mathrm{T}} + B_{e0}R^{-1}B_{ei}^{\mathrm{T}})PM_i = \Gamma_i \tag{21}$$

where, Λ_i and Γ_i are diagonal matrices which have eigenvalues on the diagonal elements. L_i and M_i are orthogonal matrices which constructed from the corresponding orthogonal vectors. From the solution P of (18), we obtain the feedback gain matrix F_e

$$F_e = -R^{-1}B_{e0}^{\mathrm{T}}P \tag{22}$$

we divide the matrix F_e into F_{e1} and F_{e2} correspond to the structure of state vector $\boldsymbol{x}(t)$.

$$\begin{aligned} u_e(t) &= -F_e\boldsymbol{x}_e(t) \\ &= -[\ F_{e1} \quad F_{e2}\ ]\begin{bmatrix} \dot{\boldsymbol{x}}(t) \\ e(t) \end{bmatrix} \\ &= -F_{e1}\dot{\boldsymbol{x}}(t) - F_{e2}e(t) \end{aligned} \tag{23}$$

To obtain input vector $u(t)$, we integrate (23).

$$\int u_e(t)dt = -\int F_{e1}\dot{\boldsymbol{x}}(t)dt - F_{e2}\int e(t)dt$$

$$u(t) = -F_{e1}\boldsymbol{x}(t) - F_{e2}\int e(t)dt \tag{24}$$

The block diagram of this system is illustrated in fig.1.

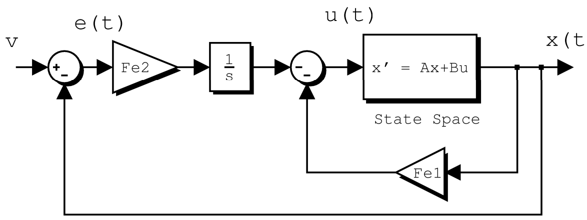

Fig.1 : Block diagram of the Integral Tracking System

4 Numerical Example

We used MATLAB software to solve the problem and simulate the system by using SIMULINK.

The stochastic algebraic Riccati equation is solved by Runge-Kutta method, standard linear quadratic regulator problem is solved by function `lqr` of control system toolbox in MATLAB.

System parameters are $m = 1, M = 1, L = 1, I = 1/3, g = 9.8$ and $\Delta\theta = 0.07$. The weighting matrices of performance index are $Q = [\ 1,\ 1,\ 0.01,\ 0.01,\ 1\ ], R = 1$. The solution of the stochastic algebraic Riccati equation (Proposed method) is

$$P = \begin{bmatrix} 801.8824 & 280.2749 & 93.5159 & 119.8140 & -28.1920 \\ 280.2749 & 99.3540 & 35.0210 & 44.7910 & -10.4630 \\ 93.5159 & 35.0210 & 21.2097 & 20.9584 & -7.7061 \\ 119.8140 & 44.7910 & 20.9584 & 25.4665 & -6.3596 \\ -28.1920 & -10.4630 & -7.7061 & -6.3596 & 5.0472 \end{bmatrix}$$

Feedback gain is

$$F_{e1} = \begin{bmatrix} -72.3137 & -23.7796 & -4.2459 & -6.5014 \end{bmatrix},$$

$$F_{e2} = \begin{bmatrix} 1.1901 \end{bmatrix}$$

Eigenvalues of the closed-loop system are

$$(\ -0.7374,\ -3.7640,\ -3.3146,\ -0.6253 \pm 0.6080i\ )$$

The solution of the algebraic Riccati equation (Ordinary method) is

$$P^* = \begin{bmatrix} 621.7728 & 213.4494 & 57.1646 & 80.2692 & -20.6857 \\ 213.4494 & 74.4296 & 21.2874 & 29.9650 & -7.6877 \\ 57.1646 & 21.2874 & 10.8762 & 12.0466 & -4.9095 \\ 80.2692 & 29.9650 & 12.0466 & 16.3369 & -4.5157 \\ -20.6857 & -7.6877 & -4.9095 & -4.5157 & 3.1351 \end{bmatrix},$$

Feedback gain is

$$F^*_{e1} = \begin{bmatrix} -63.8542 & -20.6857 & -3.1351 & -4.9095 \end{bmatrix},$$

$$F^*_{e2} = \begin{bmatrix} 1.0000 \end{bmatrix}$$

Eigenvalues of the closed-loop system are

$$(\ -3.7314,\ -3.1562,\ -0.7844,\ -0.4059 \pm 0.6868i\ )$$

Table 1: Comparison of the results

	Ordinary	Proposed
Overshoot [%]	7	0.5
Delay Time [sec]	5.28	5.42

For these two cases, we simulate the disturbed system. In time interval $[0, 3)$, the reference is $v = 0$, In $[3, 20)$, $v = 0.1$ and $[20, -40)$, $v = 0$. Overshoot and delay time of the position of the car are in Table 1. The simulation result shows that our proposed method barely increase the delay time (1.02% increased), but widely reduce the overshoot (93.9% decreased).

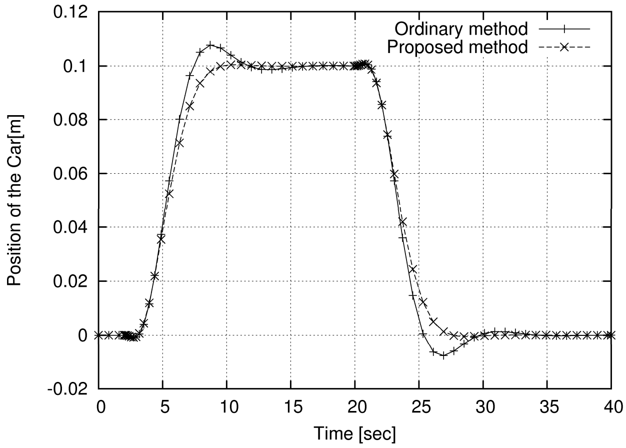

Fig.2 : Simulation results

5 Conclusion

In this paper, we considered the effect of the uncertainty in rotation angle and proposed the design method for the robust inverted pendulum-car system model with structured uncertainty. We extended this system to the integral tracking control system and applied guaranteed cost control design method to obtain the robust stable tracking control system. Through the numerical example, we showed the effectiveness of our proposed method. Future study is to apply our method for more complex model, e.g. double inverted pendulum system.

References

[1] S. Yamamoto, Q. Shen and H. Kimura, "Quadratic stabilization for a class of uncertain linear systems and its application to a magnetic levitation system", *Trans. of the Society of Instrument and Control Engineers*, Vol.29, pp.334-339, 1993.

[2] N. Takahashi, M. Kono, K. Hiranuma and O. Sato, "Generalization of guaranteed cost control", *Trans. of the Japan Society of Mechanical Engineers (Series C)*, Vol.66, No.645, pp.1531-1536, 2000.

[3] H. Kimura, T. Fujii and T. Mori, Robust Control, Corona, 1994.

[4] T. Kimura, M. Kono, N. Takahashi and M. Ishitobi, "Guaranteed cost control of tracking problem in a helicopter", *The 48th annual conference of Joint Automatic Control Conference*, pp. 241-244, 2005.