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On the Observability and Estimability Analysis of the Global Positioning System (GPS) and Inertial Navigation System (INS)

Man Hyung Lee *, Sinpyo Hong **, Jeong Hyun Moon***, Ho-Hwan Chun****,

* School of Mechanical Engineering, Pusan National University, Busan 609-735, Korea
** General Marine Business Inc, Ulsan 680-819, Korea
*** Graduate School of Mechanical Engineering, Pusan National University,
Busan 609-735, Korea (Tel : 82-51-510-1456; Fax : 82-51-512-9835)
**** Advanced Ship Engineering Research Center, Pusan National University, Busan 609-735, Korea

Abstract In this paper a brief review on the observability and estimability analysis of GPS/INS is presented. There have been various analysis results on the observability of INS errors. However different INS error dynamics models and reference frames of INS mechanization have been used in the observability analysis. Moreover, the analysis framework was not unique. In this paper, known observability analysis results are summarized first. Then relatively general analysis tools to handle system model perturbation on the observability and estimability is given.

Keywords Observability, estimability, measure, GPS, INS

1. Introduction

The observability of measurement systems has been studied to understand the behavior of error estimators in the aided inertial navigation systems (INS) [1, 2, 3, 4, 5]. For the observability study of INS, error covariance matrices or observability matrices have usually been investigated. Error covariance has been considered to be related to the degree of observability in the Kalman filter applications [6]. The error covariance test is also an efficient means of statistical study on the behavior of estimators [7]. In the analytic studies, the rank of the observability matrix was mainly investigated [3, 8, 9, 10, 11, 12]. Analytical approaches to the observability study provide insights on the estimator behavior in a more systematic way.

In the error covariance test, error covariance matrices are considered to give useful information on the degree of observability. However, the behavior of error covariance can be sensitive to the initial error covariance and the relation between the observability and the error covariance can be misleading [13]. On the other hand, in the rank test on the observability matrix, the rank does not provide the degree of observability. It only decides if a system is observable or not. In addition, the rank of a matrix can be sensitive to perturbation. Moreover, it is usually quite difficult to decide the rank of an observability matrix analytically except for very simple system models [3, 9, 10, 11, 12]. To study the degree of observability analytically, several measures of observability have been proposed. Frequency domain observability measures for time invariant systems were suggested in [14, 15]. The condition number of the observability matrix for single-output time-varying systems was considered as a measure of observability in [16]. However, these measures are not suitable for time-varying multi-input/multi-output systems. In this paper, observability measures are considered for a wider class of systems.

Measures of observability for a wider class of systems were introduced to study the degree of observability in [17]. The measure indicates how far, in the sense of 2-norm, the information matrix is from rank-deficient matrices. The measures are less sensitive to perturbation and applicable to time-varying multi-input/multi-output systems.

The concept of estimability was introduced to characterize the behavior of state estimation in [18]. Estimability measures were also introduced to indicate the ratio of error covariance decrease to the initial error covariance in [17]. It was shown that the sensitivity of the measures to perturbation depends on the size of initial error covariance.

2. Observability Analysis of GPS/INS

An INS consists of 3-axis accelerometers, 3-axis gyros, and a computer. The computer calculates position, velocity, and attitude by integrating the inertial sensor measurements. Due to the initial errors and sensor measurement uncertainty, the results of computer calculation contain errors that grow as time elapses.

If a position or velocity measurement is taken, some of the errors can be eliminated or estimated. The error that can be estimated depends upon the motion of a vehicle in which the INS is installed. The effect of the measurement on the navigation state error

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estimation has usually been studied with the behavior of error covariance matrix in the Kalman filter [6].

The first control theoretic study on the observability analysis on the INS errors was attempted by an piece-wise constant systems modeling in [3]. For the vehicle that moves with a constant speed on the horizontal plane with velocity measurement, the paper showed that the three components of attitude are unobservable. The paper also showed that the components of errors in attitude and gyro bias that are orthogonal to the direction of the acceleration change are made observable.

A more detailed observability analysis on the aided INS is given in [11]. In the paper, estimation of INS errors and the lever arm between the inertial sensors and the GPS antenna is studied with a simplified INS error dynamics model. For a vehicle with the horizontal constant speed motion, it is shown that the attitude as well as the lever and the vertical component of gyro bias are unobservable. It is also shown that acceleration changes enhance the estimates of attitude and gyro bias. The components of errors in attitude and gyro bias that are orthogonal to the direction of the acceleration change are made observable. The changes in angular rate also improve the estimate of the lever arm.

The vertical component of gyro bias is known to be nearly unobservable or weakly observable from experiences. It is interesting to note that the observability analysis with the simplified INS error dynamics model in [11] confirms the experience. However, in the observability study with relatively exact INS error dynamics model [3], the vertical component of gyro bias is observable.

3. Observability and Estimability Measures

In this section, observability and estimability measures and their properties in [17, 19] are introduced. With the measures, sensitivity of observability and estimability to perturbation in the system model can be analyzed. The influence of the initial error covariance on the error covariance is studied in detail.

Consider the following system:

$$x_i = \Phi_{i,0} x_0 \tag{1}$$

$$y_i = H_i x_i + v_i \tag{2}$$

where $x_i \in \mathbb{R}^n$ is the state vector at the time step *i*, $x_0 \in \mathbb{R}^n$ is the initial state vector, $\Phi_{i,0} \in \mathbb{R}^{n\times n}$ is the state transition matrix from the time step 0 to the time step *i*, $y_i \in \mathbb{R}^m$ is the measurement vector at the time step *i*, $v_i \in \mathbb{R}^m$ is the measurement noise vector at the time step *i*, and $H_i \in \mathbb{R}^{m\times n}$ is the measurement matrix at the time step *i*. Assume that $x_0 \sim N(\bar{x}_0, P_0)$ with $P_0 > 0$, $v_i \sim N(0, R_i)$ with $R_i > 0$ for $i = 1, 2, \cdots$, $E\left[v_i v_j^T\right] = 0$ for $i \neq j$, and $E\left[v_i x_0^T\right] = 0$ for all *i*. The optimal estimation problem considered in this section is as follows: Given a set of measurements $\{y_0, y_1, \cdots, y_k\}$ find the optimal estimate of x_0, $\hat{x}_{0,k}$, that minimizes the cost function

$$J = \frac{1}{2}\left\{ \left(x_0 - \bar{x}_0\right)^T P_0^{-1}\left(x_0 - \bar{x}_0\right) + \sum_{i=0}^{k}\left(y_i - H_i x_i\right)^T R_i^{-1}\left(y_i - H_i x_i\right) \right\}$$

The above weighted-least-squares estimate is identical to the conditional expected-value estimate that is also the minimum variance or maximum-likelihood estimate [20,21]. The optimal estimate is given as

$$\hat{x}_{0,k} = \left(P_0^{-1} + L_{0,k}\right)^{-1}\left(K_{0,k} + P_0^{-1}\bar{x}_0\right) \tag{3}$$

with

$$L_{0,k} = \sum_{i=0}^{k} \Phi_{i,0}^T H_i^T R_i^{-1} H_i \Phi_{i,0},\ K_{0,k} = \sum_{i=0}^{k} \Phi_{i,0}^T H_i^T R_i^{-1} y_i \tag{4}$$

where $L_{0,k}$ is the observability gramian or information matrix. Note that if the measurements do not have noise, then the system is deterministic and the corresponding observability gramian is

$$\mathcal{L}_{0,k} = \sum_{i=0}^{k} \Phi_{i,0}^T H_i^T H_i \Phi_{i,0} \tag{5}$$

Thus, observability gramians for stochastic and deterministic systems differ only in scaling due to R_i. Therefore, a stochastic system is observable if and only if the corresponding deterministic system is observable. The above stochastic system is observable on [0,*k*] if and only if $L_{0,k} > 0$ [22]. If the system is unobservable, then a vector x_u, called an unobservable state, exists such that $L_{0,k} x_u = 0$. The null space of $L_{0,k}$ is called the unobservable subspace.

Let $\tilde{x}_{0,k} = \hat{x}_{0,k} - x_0$. Then, the error covariance matrix is defined as

$$P_{0,k} \triangleq E\left[\tilde{x}_{0,k}\tilde{x}_{0,k}^T\right] \tag{6}$$

Then, we have

$$\left(P_{0,k}\right)^{-1} = P_0^{-1} + L_{0,k} \tag{7}$$

Since

$$\left(P_0 - P_{0,k}\right)\left(P_0\right)^{-1} = P_{0,k} L_{0,k} \tag{8}$$

the null spaces for $L_{0,k}$ and $\left(P_0 - P_{0,k}\right)$ can be different. Thus, the error covariance of an unobservable state can experience a decrease

For the observability study, consider the following measure for $M, \Delta \in \mathbb{R}^{r\times s}, r \geq s$:

$$\underline{\mu}(M) \triangleq \min_{rank(M-\Delta)<s} \|\Delta\|_2 \tag{9}$$

It indicates the magnitude of the smallest perturbation in *M* that makes *M* rank-deficient. For a matrix $M \in \mathbb{R}^{r\times s}$ with $r \geq s$, $\sigma_1(M), \sigma_1(M), \cdots, \sigma_s(M)$ denote the singular values of *M* such that $\sigma_1(M) \geq \sigma_2(M) \geq \cdots \geq \sigma_s(M) \geq 0$. Let $\bar{\sigma}(\bullet)$ and $\underline{\sigma}(\bullet)$ be the largest and the smallest singular values of a matrix, respectively. Then the following theorem essentially comes from Theorem 2.5.3 in [23]:

Theorem 1: Let $M \in \mathbb{R}^{r\times s}, r \geq s$. Then $\underline{\mu}(M) = \underline{\sigma}(M)$.

It is well-known that singular values of a matrix are well-conditioned to perturbation such that

$$\left|\sigma_i(M+E) - \sigma_i(M)\right| \leq \bar{\sigma}(E) \tag{10}$$

for $i = 1, 2, \cdots, s$, and $E \in \mathbb{R}^{r \times s}$ [23,24] So, if we define an observability measure for a system with $\underline{\mu}(L_{0,k})$, then $\underline{\sigma}(L_{0,k})$ indicates how far the system is from the rank-deficient matrices. The measure is less sensitive to perturbation due to errors in the system model or the numerical computation.

In many estimation applications, the measure of observability for a subspace can be quite convenient to predict or understand the behavior of the subspace of the state-space. For this purpose, the following norm can be useful to define the measure:

$$\mu(M, z) \triangleq \min_{(M-\Delta)z=0} \|\Delta\|_2 \tag{11}$$

where $M, \Delta \in \mathbb{R}^{r \times s}, r \ge s$ and $z \in \mathbb{R}^s$. With the definition, we have the following theorems:

Theorem 2: Let $M \in \mathbb{R}^{r \times s}, r \ge s, z \in \mathbb{R}^s$ with $\|z\|_2 = 1$. Then

$$\mu(M, z) = \|Mz\|_2 \tag{12}$$

Proof: See [17].

Theorem 3: Let $M, E \in \mathbb{R}^{r \times s}, r \ge s$ and $z \in \mathbb{R}^s$ with $\|z\|_2 = 1$. Then,

$$\left|\mu(M + E, z) - \mu(M, z)\right| \le \bar{\sigma}(E) \tag{13}$$

Proof: See [17]

Theorem 3 shows that $\mu(M, z)$ is also well-conditioned to the perturbation in M.

The other observability measure can be defined with $\mu(L_{0,k}, z)$. This measure indicates the magnitude of the smallest perturbation in the information matrix that makes the subspace spanned by the vector z unobservable. Let the singular value decomposition (SVD) of $L_{0,k}$ be $U_k \Sigma_k U_k^T$ where $U_k = [u_1 u_2 \cdots u_n]$ is an orthogonal matrix composed of singular vectors and $\Sigma_k = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$ is a diagonal matrix whose diagonal elements are the singular values of $L_{0,k}$ such that $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_n \ge 0$. Since $\mu(L_{0,k}, z) = \|\Sigma_k U_k^T z\|_2$,

$$\mu(L_{0,k}, u_i) = \sigma_i(L_{0,k}) \tag{14}$$

Therefore, a singular value of the information matrix can be considered as the measure of observability for the subspace spanned by the corresponding singular vector. A large singular value implies that a large change in the information matrix is necessary to make the subspace spanned by the corresponding singular vector unobservable. It is apparent that the system observability measure is the smallest subspace observability measure over the whole state-space. Thus, we have the following relation:

$$\underline{\mu}(L_{0,k}) = \min_{z \in \mathbb{R}^n} \mu(L_{0,k}, z) \tag{15}$$

In estimation applications, the behavior of the error covariance is one of the main concerns of estimator designers. To characterize estimator performance, the term 'estimability' will be used. A system is called estimable if $P_0 - P_{0,k} > 0$. The null space of $P_0 - P_{0,k}$ is referred to as an unestimable subspace. Then, a measure of estimability for a subspace may be defined with

$$\nu(L_{0,k}, P_0, u) = \frac{u^T (P_0 - P_{0,k}) u}{u^T P_0 u} \tag{16}$$

for $u \in \mathbb{R}^n$. It indicates the ratio of the decrease in the error covariance of a state in the direction of u to the initial error covariance of the same state. The concept of estimability that is used in this paper is similar to that in [18]. However, estimability in [18] is concerned with the error covariance rather than the initial error covariance. The connection between the observability and estimability can be found in (8). From this equation it can be shown that a system is observable if and only if the system is estimable. If the span of a vector u is unobservable, then $\text{span}\{P_0^{-1} u\}$ is unestimable. Then, the following theorem shows the the estimability measure is less sensitive to perturbation in the information matrix if the magnitude of the initial error covariance matrix is not excessively large:

Theorem 4: Let $E \in \mathbb{R}^{n \times n}$, $r \triangleq 1 + \underline{\sigma}(L_{0,k}) \underline{\sigma}(P_0) > \bar{\sigma}(E)$, and $u \in \mathbb{R}^n$. Then,

$$\left|\nu(L_{0,k} + E, P_0, u) - \nu(L_{0,k}, P_0, u)\right| \le \frac{\bar{\sigma}(E) \bar{\sigma}(P_0)}{r(r - \bar{\sigma}(E))} \tag{17}$$

Proof: See [19].

Let the SVD of $\sqrt{P_0} L_{0,k} \sqrt{P_0}$ be $U_{pl} \Sigma_{pl} U_{pl}^T$ where Σ_{pl} is $\text{diag}(\sigma_{pl,1}, \sigma_{pl,2}, \cdots, \sigma_{pl,n})$ and $U_{pl} = [u_{pl,1} u_{pl,2} \cdots u_{pl,n}]$. Let $d_{pl,i} = \sigma_{pl,i} / (1 + \sigma_{pl,i})$, i=1,2,$\cdots$,$n$. Then,

$$\nu(L_{0,k}, P_0, u) = \frac{u^T \sqrt{P_0} U_{pl} D_{pl} U_{pl}^T \sqrt{P_0} u}{u^T P_0 u} \tag{18}$$

where $D_{pl} = \text{diag}(d_{pl,1}, d_{pl,2}, \cdots, d_{pl,n})$. Thus, SVD of $\sqrt{P_0} L_{0,k} \sqrt{P_0}$ gives useful information on the estimability measure.

4. Conclusions

The exact model of INS error dynamics is nonlinear and highly complicated. Observability study of GPS/INS usually involves simplification of the INS error model. The observability analysis by rank test is sensitive to the system model perturbation. Observability anaysis of GPS/INS by covariance analysis can be statistically convenient. However, the error covariance behavior is highly influenced by the choice of initial error covariance and can give misleading results on the observability.

With the proposed observability and estimability measures, a straight forward analysis on the characteristics of the observability and estimability of GPS/INS error is possible. The observability analysis results are insensitive to system model perturbation. The estimability measures show that the sensitivity of error covariance to system model perturbation can be influenced by the choice of the initial error covariance.

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