The Fourteenth International Symposium on Artificial Life and Robotics 2009 (AROB 14th '09),
B-Con Plaza, Beppu, Oita, Japan, February 5 - 7, 2009

Robust stability analysis for uncertain T–S fuzzy systems with a time-varying delay

In-sun Song, Jae Wook Shin, Jeong-Wan Ko and PooGyeon Park
{weedsis, shinshingo, hrp26, ppg}@postech.ac.kr

Electrical and Computer Engineering Division,
Pohang University of Science and Technology,
San 31, Hyojadong, Namgu, Pohang, Kyungbuk, 790-784, Korea

Abstract

In this paper, we adopt fuzzy weighting-dependent free variables in the system dynamics elimination and fuzzy weighting-dependent kernel matrices in the integral inequality lemma to maximize the allowable delay bounds that guarantee the stability of Takagi-Sugeno (T–S) fuzzy systems with time-varying delays. The resulting quadratic *Parameterized Linear Matrix Inequalities* (PLMIs) are further relaxed by introducing some free variables for the weighting parameters conditions itself. A simple example is given to demonstrate the effectiveness of the proposed criterion.

keywords

Fuzzy delayed systems, Uncertainty, Constraint elimination, Stability.

1 Introduction

T–S fuzzy systems, characterized by linearity of the local dynamics in different fuzzy sets of fuzzy rules, have been studied widely and applied to various fields of industrial applications [1–5]. Among many attracting topics, we shall focus on the stability analysis of the fuzzy system that has delay elements in the local dynamics.

A general continuous T–S fuzzy control system with a time-varying delay can be described as

$$\dot{x}(t) = A(t)x(t) + A_h(t)x(t-h(t)) + B(t)u(t) \tag{1}$$

$$= \sum_{i=1}^{r} \theta_i(t)\{A_i x(t) + A_{h,i}x(t-h(t)) + B_i u(t)\}, \tag{2}$$

$$\sum_{i=1}^{r} \theta_i(t) = 1 \ \text{ and } \ \theta_i(t) \geq 0, \ i \in \{1, \cdots, r\}, \tag{3}$$

where $\{\theta_i(t)\}$ may depend on some premise variables. Up to now, the usual stability criteria have been represented by way of PLMIs with an affine dependence on $\{\theta_i(t)\}$. To make use of the efficient convex optimization tools, the structure of the decision variables that are multiplied by $\{A(t), A_h(t), B(t)\}$ has been restricted to be independent from the fuzzy weighting functions. See, for example, the free-weighting matrices in the constraint elimination method [1–3] and the kernel matrices in the quadratic Lyapunov function [1–5], etc. Recently, [6, 7] reported the conservativeness of these methods and suggested a relaxation method of handling PLMIs that are not affinely dependent on $\{\theta_i(t)\}$. The resulting nonlinear parameter conditions, usually quadratic functions, could be properly relaxed but the information of (3) could be preserved faithfully.

In this paper, we adopt these constraint relaxation techniques to maximize the allowable delay bounds that guarantee the stability of the T–S fuzzy systems. Both the free variables in the constraint elimination of (1)–(2) [3, 8–10] and the kernel matrices in the integral inequality lemma [11, 12] are modeled as fuzzy weighting-dependent functions. The resulting quadratic PLMIs are further relaxed by introducing some free variables for the weighting parameter conditions in (3). Our approach essentially reduces the conservatism of the existing methods.

The paper is organized as follows. Section 2 will consider a robust stability criterion for T–S fuzzy systems with time-varying delays. Section 3 will show a simple example for verification of the criterion.

2 Main Results

Let us consider the following T–S fuzzy delayed system with model uncertainty:

$$\dot{x}(t) = A(t)x(t) + A_h(t)x(t-h(t)) + Dp(t), \ t \geq 0, \tag{4}$$

$$q(t) = E(t)x(t) + E_h(t)x(t-h(t)), \ t \geq 0, \tag{5}$$

$$x(t) = \phi(t), \ -\bar{h} \leq t \leq 0, \tag{6}$$

$$[A(t) \ A_h(t) \ E(t) \ E_h(t)] = \sum_{i=1}^{r} \theta_i(z(t))[A_i \ A_{h,i} \ E_i \ E_{h,i}], \tag{7}$$

where $h(t) \in [0, \ \bar{h}]$, $p(t) = \Delta(t)q(t)$, $\Delta^T(t)\Delta(t) \leq$

$\gamma^{-2}I$, and $\phi(t) \in \mathcal{C}^1(\bar{h})$, the set of continuously differentiable functions in the domain $[-2\bar{h}, 0]$. Here, $\{\theta_i(z(t))\}$ denote the normalized fuzzy weighting functions that satisfy

$$0 \leq \theta_i(z(t)) \leq 1, \ \forall\, i, \text{ and } \sum_{i=1}^{r} \theta_i(z(t)) = 1, \quad (8)$$

where r is the number of fuzzy rules and $z(t)$ is a premise variable vector that may depend on states in many cases. Let us define $\chi(t) \in \mathcal{R}^{l\times 1}$ as $\chi(t) \triangleq [x^T(t)\ x^T(t-h(t))\ x^T(t-\bar{h})\ \dot{x}^T(t)\ p^T(t)]^T$ and the corresponding block entry matrices as $e_i,\ i \in \{1, \cdots, 5\}$ such that the system (4) can be written as $0 = (A(t)e_1^T + A_h(t)e_2^T - e_4^T + De_5^T)\chi(t)$.

We shall choose the Lyapunov-Krasovskii functional as follows:

$$V(t) \triangleq V_1(t) + V_2(t) + V_3(t), \quad (9)$$

$$V_1(t) = x^T(t)Px(t),\ P > 0, \quad (10)$$

$$V_2(t) = \int_{t-\bar{h}}^{t} x^T(\alpha)Q_0x(\alpha)d\alpha,\ Q_0 > 0, \quad (11)$$

$$V_3(t) = \int_{-\bar{h}}^{0}\int_{t+\alpha}^{t} \dot{x}^T(\beta)S_0\dot{x}(\beta)d\beta d\alpha,\ S_0 > 0 \quad (12)$$

such that

$$\dot{V}_1(t) = 2\dot{x}^T(t)Px(t) = 2\chi^T(t)e_4Pe_1^T\chi(t), \quad (13)$$

$$\dot{V}_2(t) = \chi^T(t)\{e_1Q_0e_1^T - e_3Q_0e_3^T\}\chi(t), \quad (14)$$

$$\dot{V}_3(t) = \bar{h}\chi^T(t)e_4S_0e_4^T\chi(t) - \int_{t-\bar{h}}^{t} \dot{x}^T(\alpha)S_0\dot{x}(\alpha)d\alpha. \quad (15)$$

Then, by the integral inequality lemma [11, 12], for

$$\begin{bmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{12}^T(t) & Y_{22}(t) \end{bmatrix} \geq 0,\ \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{12}^T(t) & Z_{22}(t) \end{bmatrix} \geq 0, \quad (16)$$

we have

$$\begin{aligned} 0 &\leq \int_{t-h(t)}^{t} \begin{bmatrix} \chi(t) \\ \dot{x}(\alpha) \end{bmatrix}^T \begin{bmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{12}^T(t) & Y_{22}(t) \end{bmatrix} \begin{bmatrix} \chi(t) \\ \dot{x}(\alpha) \end{bmatrix} d\alpha \\ &= \chi^T(t)\{Y_{12}(t)(e_1 - e_2)^T + (e_1 - e_2)Y_{12}^T(t) \\ &\quad + h(t)Y_{11}(t)\}\chi(t) + \int_{t-h(t)}^{t} \dot{x}^T(\alpha)Y_{22}(t)\dot{x}(\alpha)d\alpha, \end{aligned}$$

$$\begin{aligned} 0 &\leq \int_{t-\bar{h}}^{t-h(t)} \begin{bmatrix} \chi(t) \\ \dot{x}(\alpha) \end{bmatrix}^T \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{12}^T(t) & Z_{22}(t) \end{bmatrix} \begin{bmatrix} \chi(t) \\ \dot{x}(\alpha) \end{bmatrix} d\alpha \\ &= \chi^T(t)\{Z_{12}(t)(e_2 - e_3)^T + (e_2 - e_3)Z_{12}^T(t) \\ &\quad + (\bar{h} - h(t))Z_{11}(t)\}\chi(t) + \int_{t-\bar{h}}^{t-h(t)} \dot{x}^T(\alpha)Z_{22}(t)\dot{x}(\alpha)d\alpha, \end{aligned}$$

so that $\dot{V}(t)$ can be upper-bounded by the following quantity:

$$\dot{V}(t) \leq \chi^T(t)\Omega_1\chi(t) + \Omega_2,$$

where Ω_i denote

$$\begin{aligned} \Omega_1 = {} & h(t)Y_{11}(t) + (\bar{h} - h(t))Z_{11}(t) + \bar{h}e_4S_0e_4^T \\ & + Y_{12}(t)(e_1 - e_2)^T + (e_1 - e_2)Y_{12}^T(t) \\ & + Z_{12}(t)(e_2 - e_3)^T + (e_2 - e_3)Z_{12}^T(t) \\ & + e_1Q_0e_1^T - e_3Q_0e_3^T + e_4Pe_1^T + e_1Pe_4^T, \end{aligned} \quad (17)$$

$$\begin{aligned} \Omega_2 = {} & -\int_{t-\bar{h}}^{t-h(t)} \dot{x}^T(\alpha)(S_0 - Z_{22}(t))\dot{x}(\alpha)d\alpha \\ & - \int_{t-h(t)}^{t} \dot{x}^T(\alpha)(S_0 - Y_{22}(t))\dot{x}(\alpha)d\alpha. \end{aligned} \quad (18)$$

Clearly, if it holds that $S_0 – Z_{22}(t) \geq 0$ and $S_0 – Y_{22}(t) \geq 0$, Ω_2 is non-positive definite, *i.e.* $\Omega_2 \leq 0$. As for Ω_1, since $h(t)Y_{11}(t) + (\bar{h} - h(t))Z_{11}(t)$ is a convex combination of the matrices $Y_{11}(t)$ and $Z_{11}(t)$ on $h(t)$, it can be handled non-conservatively by two corresponding boundary LMIs: one for $h(t) = \bar{h}$ and the other for $h(t) = 0$. Furthermore, we can remove the constraints of the model dynamics itself in (4) by introducing free variables $M(t)$ as $0 \equiv \chi^T(t)M(t)(A(t)e_1^T + A_h(t)e_2^T - e_4^T + De_5^T)\chi(t)$ like [8–10], and the additional uncertainty constraint (5):

$$\begin{aligned} 0 &\leq q^T(t)q(t) - \gamma^2 p^T(t)p(t) \\ &= \chi^T(t)\{(e_1E^T(t) + e_2E_h^T(t))(E(t)e_1^T + E_h(t)e_2^T) \\ &\quad - \gamma^2 e_5e_5^T\}\chi(t), \end{aligned} \quad (19)$$

which can be handled through the so called $\mathcal{S}$-procedure [13]. Then, we can state the following theorem for robust stability of the delayed T–S fuzzy system.

Theorem 1 *For a given γ, the delayed uncertain T–S fuzzy system (4)–(7) is asymptotically stable if there exist matrices P, Q_0, S_0, $Y_{11}(t)$, $Y_{12}(t)$, $Y_{22}(t)$, $Z_{11}(t)$, $Z_{12}(t)$, $Z_{22}(t)$, and $M(t)$ such that the following conditions hold:*

$$P > 0, Q_0 > 0, S_0 > 0, S_0 \geq Z_{22}(t), S_0 \geq Y_{22}(t), \quad (20)$$

$$\begin{bmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{12}^T(t) & Y_{22}(t) \end{bmatrix} \geq 0, \begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{12}^T(t) & Z_{22}(t) \end{bmatrix} \geq 0, \quad (21)$$

$$\begin{aligned} 0 > {} & M(t)(A(t)e_1^T + A_h(t)e_2^T - e_4^T + De_5^T) \\ & + (e_1A^T(t) + e_2A_h^T(t) - e_4 + e_5D^T)M^T(t) \\ & + (e_1E^T(t) + e_2E_h^T(t))(E(t)e_1^T + E_h(t)e_2^T) \\ & + Y_{12}(t)(e_1 - e_2)^T + (e_1 - e_2)Y_{12}^T(t) \\ & + Z_{12}(t)(e_2 - e_3)^T + (e_2 - e_3)Z_{12}^T(t) \\ & + \bar{h}e_4S_0e_4^T + e_1Q_0e_1^T - e_3Q_0e_3^T + e_4Pe_1^T + e_1Pe_4^T \\ & + \bar{h}(\delta(1,k)Y_{11}(t) + \delta(2,k)Z_{11}(t)) - \gamma^2 e_5e_5^T \end{aligned} \quad (22)$$

for $k = 1, 2$, where $\delta(i, i) = 1$ and $\delta(i, j) = 0,\ i \neq j$.

Let us model both the free variables employed for the constraint elimination of (4) [3, 8–10] and the kernel matrices in the integral inequality lemma for the double integral terms of the Lyapunov-Krasovskii functionals [11, 12] as fuzzy weighting $\{\theta_i(z(t))\}$-dependent functions:

$$M(t) = \sum_{i=1}^{r} \theta_i(z(t)) M_i, \tag{23}$$

$$\begin{bmatrix} Y_{11}(t) & Y_{12}(t) \\ Y_{12}^T(t) & Y_{22}(t) \end{bmatrix} \triangleq \sum_{i=1}^{r} \theta_i(z(t)) \begin{bmatrix} Y_{11,i} & Y_{12,i} \\ Y_{12,i}^T & Y_{22,i} \end{bmatrix}, \tag{24}$$

$$\begin{bmatrix} Z_{11}(t) & Z_{12}(t) \\ Z_{12}^T(t) & Z_{22}(t) \end{bmatrix} \triangleq \sum_{i=1}^{r} \theta_i(z(t)) \begin{bmatrix} Z_{11,i} & Z_{12,i} \\ Z_{12,i}^T & Z_{22,i} \end{bmatrix}, \tag{25}$$

$$\begin{bmatrix} Y_{11,i} & Y_{12,i} \\ Y_{12,i}^T & Y_{22,i} \end{bmatrix} \geq 0, \begin{bmatrix} Z_{11,i} & Z_{12,i} \\ Z_{12,i}^T & Z_{22,i} \end{bmatrix} \geq 0, \ \forall\ i. \tag{26}$$

Unfolding the $\{\theta_i(z(t))\}$-dependent time-varying matrices, we can rewrite (22) as a quadratic function of $\Theta(t) \triangleq [I\ \theta_1(z(t))I\ \cdots\ \theta_r(z(t))I]^T \in \mathcal{R}^{(r+1)\cdot l \times l}$:

$$0 > \Theta^T(t)\Omega_{1,k}\Theta(t), \quad k = 1, 2, \tag{27}$$

where $\Omega_{1,k}$ are $(r+1) \times (r+1)$-block time-invariant symmetric matrices. Finally, the constraints elimination method [6, 7] for (8) gives

$$\Theta^T(t)N_{0,k}F\Theta(t) + \Theta^T(t)F^TN_{0,k}^T\Theta(t) = 0, \tag{28}$$

$$F \triangleq [-I\ I\ I\ \cdots\ I] \in \mathcal{R}^{l\times(r+1)\cdot l} \tag{29}$$

for arbitrary $N_{0,k} \in \mathcal{R}^{(r+1)\cdot l \times l}$ and

$$\theta_i(z(t))\theta_i(z(t))N_{1,k,i} \geq 0, \tag{30}$$

$$\theta_i(z(t))\theta_j(z(t))(N_{2,k,ij} + N_{2,k,ij}^T) \geq 0, \tag{31}$$

$$\theta_i(z(t))(1 - \theta_i(z(t)))(N_{3,k,i} + N_{3,k,i}^T) \geq 0 \tag{32}$$

for

$$N_{1,k,i} \geq 0,\ N_{2,k,ij} + N_{2,k,ij}^T \geq 0,\ N_{3,k,i} + N_{3,k,i}^T \geq 0, \tag{33}$$

where $1 \leq i \leq r$, $i < j \leq r$ and $k = 1, 2$. Then, we can obtain the following PLMIs-based stability criterion for delayed T–S fuzzy systems.

Theorem 2 *For a given γ, the delayed uncertain T–S fuzzy system (4)–(7) is asymptotically stable if there exist matrices P, Q_0, S_0, $Y_{11,i}$, $Y_{12,i}$, $Y_{22,i}$, $Z_{11,i}$, $Z_{12,i}$, $Z_{22,i}$, M_i, $N_{0,k}$, $N_{1,k,i}$, $N_{2,k,ij}$ and $N_{3,k,i}$, $1 \leq i \leq r$, $i < j \leq r$, $k = 1, 2$ such that the following conditions hold:*

$$P > 0, Q_0 > 0, S_0 > 0, S_0 \geq Z_{22,i}, S_0 \geq Y_{22,i}, \tag{34}$$

$$0 > \Omega_{1,k} + N_{0,k}F + F^TN_{0,k}^T, \tag{35}$$

$$\begin{bmatrix} Y_{11,i} & Y_{12,i} \\ Y_{12,i}^T & Y_{22,i} \end{bmatrix} \geq 0, \begin{bmatrix} Z_{11,i} & Z_{12,i} \\ Z_{12,i}^T & Z_{22,i} \end{bmatrix} \geq 0, \tag{36}$$

$$N_{1,k,i} \geq 0, N_{2,k,ij} + N_{2,k,ij}^T \geq 0, N_{3,k,i} + N_{3,k,i}^T \geq 0, \tag{37}$$

where $\Omega_{1,k}$ are $(r+1)\times(r+1)$-block symmetric matrices whose elements are

$$\begin{aligned}
\Omega_{1,k}(1,1) =& e_4Pe_1^T + e_1Pe_4^T + e_1Q_0e_1^T - e_3Q_0e_3^T \\
&+ \bar{h}e_4S_0e_4^T - \gamma^2 e_5e_5^T, \\
\Omega_{1,k}(1,i+1) =& (e_5D^T - e_4)M_i^T + N_{3,k,i} \\
&+ \frac{\bar{h}}{2}(\delta(1,k)Y_{11,i} + \delta(2,k)Z_{11,i}) \\
&+ (e_1 - e_2)Y_{12,i}^T + (e_2 - e_3)Z_{12,i}^T, \\
\Omega_{1,k}(i+1,i+1) =& M_i(A_ie_1^T + A_{h,i}e_2^T) + (e_1A_i^T + e_2A_{h,i}^T)M_i^T \\
&+ (e_1E_i^T + e_2E_{h,i}^T)(E_ie_1^T + E_{h,i}e_2^T) \\
&+ N_{1,k,i} - N_{3,k,i} - N_{3,k,i}^T, \\
\Omega_{1,k}(i+1,j+1) =& M_i(A_je_1^T + A_{h,j}e_2^T) + (e_1A_i^T + e_2A_{h,i}^T)M_j^T \\
&+ (e_1E_i^T + e_2E_{h,i}^T)(E_je_1^T + E_{h,j}e_2^T) + N_{2,k,ij}.
\end{aligned}$$

3 Examples

Example 1 Consider the uncertain T–S fuzzy delayed system (4)–(7) with two fuzzy rules. The following parameters are used.

$$A_1 = \begin{bmatrix} -1.0 & 0.4 \\ 0.0 & -0.5 \end{bmatrix},\ A_{h,1} = \begin{bmatrix} 0.3 & -0.4 \\ 0.0 & 0.0 \end{bmatrix}, \tag{38}$$

$$E_1 = \begin{bmatrix} -0.2 & 0.0 \\ 0.0 & 0.3 \end{bmatrix},\ E_{h,1} = \begin{bmatrix} 0.3 & 0.0 \\ 0.0 & 0.0 \end{bmatrix}, \tag{39}$$

$$A_2 = \begin{bmatrix} -0.5 & 0.0 \\ 0.5 & -1.0 \end{bmatrix},\ A_{h,2} = \begin{bmatrix} 0.4 & 0.0 \\ 0.4 & 0.3 \end{bmatrix}, \tag{40}$$

$$E_2 = \begin{bmatrix} 0.0 & 0.3 \\ 0.0 & 0.1 \end{bmatrix},\ E_{h,2} = \begin{bmatrix} 0.1 & -0.1 \\ 0.0 & 0.0 \end{bmatrix}, \tag{41}$$

$$D = \begin{bmatrix} 0.1 & 0.0 \\ 0.0 & 1.5 \end{bmatrix},\ \gamma = 1. \tag{42}$$

By Theorem 2, the improvement of this paper is shown in Table 1.

Table 1: maximum $\bar{h}$ comparison with some previous results

methods	[1]	[2]	[3]	[4]	[5]	Theorem 2
$\bar{h}$	0.639	0.639	0.801	0.808	0.836	1.328

4 Conclusion

In this paper, we adopted fuzzy weighting-dependent free variables in the system dynamics elimination and fuzzy weighting-dependent kernel matrices in the integral inequality lemma to maximize the allowable delay bounds that guarantee the stability of T–S fuzzy systems with time-varying delays. The resulting quadratic PLMIs were further relaxed by introducing some free variables for the weighting parameters conditions itself. A simple example was given to demonstrate the effectiveness of the proposed criterion.

Acknowledgements

This work was supported by the Brain Korea 21 Project in 2008. This research was supported by Ministry of Knowledge and Economy, Republic of Korea, under the ITRC (Information Technology Research Center) support program supervised by IITA(Institute for Information Technology Advancement)." (IITA-2008-C1090-0801-0004)

References

[1] Lien, C., Yu, K., Chen, W., Wan, Z., and Chung, Y. (2007) *IET Control Theory & Applications* **1(3)**, 764–769.

[2] Lien, C.-H. (2006) *Chaos, Solitons and Fractals* **28(2)**, 422–427.

[3] Chen, B., Liu, X., Lin, C., and Liu, K. (2008) *Fuzzy Sets and Systems* available online.

[4] Peng, C., Tian, Y.-C., and Tian, E. (2008) *Fuzzy Sets and Systems* **159(20)**, 2713–2729.

[5] Liu, X. (2008) *Nonlinear Analysis* **68(5)**, 1352–1361.

[6] Park, P. and Choi, D. J. (2001) *International Journal of Robust and Nonlinear Control* **11(14)**, 1343–1363.

[7] Choi, D. J. and Park, P. (2003) *IEEE Transactions on Fuzzy Systems* **11(2)**, 271–278.

[8] He, Y., Wu, M., She, J.-H., and Liu, G.-P. (2004) *Systems & Control Letters* **51(1)**, 57–65.

[9] He, Y., Wu, M., She, J.-H., and Liu, G.-P. (2004) *IEEE Transactions on Automatic Control* **49(5)**, 828–832.

[10] Wu, M., He, Y., She, J.-H., and Liu, G.-P. (2004) *Automatica* **40(8)**, 1435–1439.

[11] Park, P. (1999) *IEEE Transactions on Automatic Control* **44(4)**, 876–877.

[12] Moon, Y. S., Park, P., Kwon, W. H., and Lee, Y. S. (2001) *International Journal of Control* **74(14)**, 1447–1455.

[13] Boyd, S., Ghaoui, L. E., Feron, E., and Balakrishnan, V. (1994) Linear Matrix Inequalities in System and Control Thoery, SIAM, Philadelphia.