Towards a Brief Proof of the Four Colour Theorem: Theorems to be used for proving the Four Colour Theorem

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Abstract: In order to prove the Four-Colour theorem (FCT) without using computer, basic definitions and theorems useful for proving the FCT are presented and a bird-eye's view of the brief proof is described and discussed. A complete proof will appear elsewhere in near future.

Keywords: Four colour conjecture, Appel & Haken's proof, complete triangulation graph, Four-faced quadrilateral, vertex-reducing complete triangulation series, necessary and sufficient condition.

I. INTRODUCTION

Francis Frederick's observation concerning map-clouring was first submitted by his younger brother, Frederic Guthrie, as a mathematical conjecture later called the *Four-Colour Problem*, to his professor, August de Morgan in 1852 [1-4]. Since then, the Four-Colour Conjecture has long been considered to be a most difficult unsolved problem until Appel and Haken's (1972, [5,6]) success in proving this conjecture by using computer. The question whether or not the Four-Colour theorem (FCT) could be proved without using computer has since been the next important problem remaining to be answered, although *Robertson et al.* (1996) have considerably simplified Appel and Haken's proof [7,8].

In this communication, theorems most possibly useful for proving the FCT with no use of computer were presented and discussed towards prospecting and achieving the final proof. A most plausible final proof of the FCT without using computer will be published elsewhere from this aspect, which is now under reviwing (Ohnishi [9,10].).

II. PRELIMINARIES: BASIC DEFINITIONS

In this section, some basic definitions useful for proving the FCT are described. "▌" denotes the end of each definition or theorem, whereas "█" denotes the end of each proof.

[Definition 2.1](Jordan curve): *Jordan curve QQ'* is defined as a portion of a closed Jordan curve, C, cut off by two different points Q and Q', which are called *end-points* of the Jordan curve QQ'. ▌

The next theorem is well known, and is described below without proof.

[**Theorem 2.1]** (internal and external domains): If C is a closed Jordan curve on S^2, then we have $S = int\ C + C + ext\ C$, where $int\ C$ and $ext\ C$ denote internal and external domains of C, respectively. Let closed internal and external domains be defined by $Int\ C = int\ C + C$, and $Ext\ C = ext\ C + C$, respecttively, then we have $S^2 = Int\ C + Ext\ C - C$. ▌
[Proof] See Ore (1967) [11]. █

[Definition 2.2] (graph, spherical graph): Graph Γ is defined as a set consisting of a finite set of vertices and a finite setof edges. *Vertex* is defined as a point, and *edge* is defined as a Jordan curve connecting and including two vertices (which are end-points) P and P'. An edge e, connecting two vertices P and P' is written as $e = [P,P']$. $<e>$ is defined by $<e> = e - P - P'$. A vertex P is called to be *adjacent* to P', if a graph Γ has an edge, $[P,P']$. If a graph G is embeddable onto a sphere S^2, G is called a spherical graph and is written as $G(S^2)$. ▌

[Definition 2.3] (valency): If a vertex P is a common end-point of different m edges, then m is called *valency* (or *degree*) of P, and is written as $m = val\ P$. ▌

[Definition 2.4] (s-cycle, s-gon, s-path): A (s-) cycle is defined by a s-vertex-graph, $C = C^s = C^s(e_{12}, e_{23}, \ldots, e_{s,1}) = P_1 + <e_{12}> + P_2 + <e_{23}> + \ldots + <e_{s-1,s}> + P_s + <e_{s,1}>$. A ($s$-)path is defined by $U(P_1,P_s) = U^s(P_1,P_s) = C^s(e_{12}, e_{23}, \ldots, e_{s,1}) - e_{s,1}$. C^s is also called s-gon (=s-hedron) (poly-gon, di-gon, tri-angle, tetrahedron=quadrilateral, pentagon, etc.). ▌

From the <u>Definitions 2.1~2.5</u>, $U(P_1,P_s)$ is a Jordan curve connecting P_1 and P_s, and a cycle is a closed Jordan curve.

[Definition 2.5] (connected graph): If a path $U(P, P')$ of a given graph Γ can be found for any pair of vertices, P and P', which belong to Γ, Γ is called *"connected graph"*. ▌

[Definition 2.6] (face): If $G(S^2)$ has a s-cycle (= s-gon), C^s, *where* $int\ C^s = \emptyset$, $Int\ C^s$ (= $int\ C^s + C^s$) is called *face* (or *s-gon face*). ▌

Thus we find $S^2 = int\ C^s + C^s + ext\ C^s$.

[Definition 2.7]((complete) triangulation): If $G(S^2)$

is a connected graph dividing S^2 into exclusively triangular faces, G is called *"complete triangulation (of S2)"*. If $G(S^2) = C^s$ satisfies *ext* $C^s = ø$, and if G divides *Int* C^s into exclusively triangular faces, G is called *"triangulation of s-gon,* C^s*"*. ■

Let P be a vertex of complete triangulation of S^2, then we easily find $val\ P \geq 2$.

[Definition 2.8] (ν-colourable): Graph Γ is called "*vertex ν-colourable*" (or simply, *ν-colorable*), if every vertex is coloured with one of the given ν colours so that any two vertices adjacent to each other are coloured with different colours. If a ν-colourable graph G is coloured with μ colours ($\mu \leq \nu$), the coloured graph is *here* called *"ν-coloured graph"*, and is written as $col^{\nu}(G)$. For an vertex $P \in G$, *"$col^{\nu}(P) = a$"* is defined for denoting that the vertex P is coloured with a, in $col^{\nu}(G)$. If a ν-colourable graph, G, is not colourable with ν - 1 colours, G is called *"ν-chromatic"*. ■

The next theorem (Theorem 2.2) is well-known [2,4,11], and is given here without describing proof.

[Theorem 2.2]Let $T(S^2)$ be an arbitrarily selected complete triangulation of S^2. The four-colour theorem (FCT) is equivalent to that "*Proposition A* is true", where *Proposition A* is given by;

Proposition A: $T(S^2)$ is vertex four-colourable. ■

[Definition 2.9](Two-faced quadrilateral, Figure 1): "*Two-faced quadrilateral* with a diagonal edge e_{13}" is defined as a subgraph of $G(S^2)$, and is given by $Q^{2f} = C^4_0 + \langle e_{13} \rangle \subseteq G$, where $C^4_0 = C^4(e_{12},e_{23}, e_{34}, e_{41})$,$e_{ij} = [P_i, P_j]$, $e_{13} \subseteq Int\ C^4_0$, and e_{13} is a Boundary edge dividing $Int\ C^4$ into two triangular faces. Q^{2f} is written as $Q^{2f} = Q^{2f}(C^4_0;e_{13})$. If Q^{2f} in $G(S^2)$ has any edge, e'_{13} or e'_{24}, satisfying $e'_{13} = [P_1,P_3] \subseteq Ext\ C^4_0$, or $e'_{24} = [P_2,P_4] \subseteq Ext\ C^4_0$, then the $Q^{2f}$is called *"incomplete quadrilateral"*, whereas it is called *"complete quadrilateral"* if there is none of such edges. *$G(S^2)$ having its subgraph $Q^{2f}_0 = Q^{2f}(C^4_0;e_{13})$* is written as $G = G(Q^{2f}_0; C^4_0)$. ■

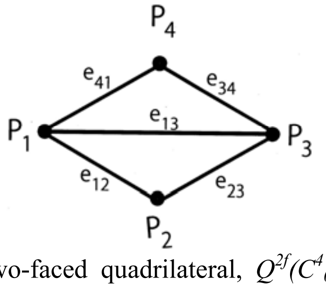

Figure 1. Two-faced quadrilateral, $Q^{2f}(C^4_0;e_{13})$, where C^4_0 is a 4-cycle (= quadrilateral) given by $C^4_0 = C^4(e_{12},e_{23},e_{34},e_{41})$, $e_{ij} = [P_i,P_j]$, $e_{13} \subseteq Int\ C^4_0$. $\{Q^{2f}\}$ is an unavoidable (one-element-)set of $T_k(S^2)$, a complete triangulation of S^2 with k vertices ($k \geq 4$). See <u>Definition 2.9</u> and <u>Lemma3.2.1</u>.

[Definition 2.10](4-coloued graph, Kempe block): Let $col^4_o(G)$ denote a 4-coloured graph of $G(S^2)$, coloured with 4 or 3 of the given 4 colours, a, b, c, and d. $col^4_o(G)$ is also called *"4-colouration of G"*. Furthermore, *ab-Kempe blocks* (= *ab-Kempe chains*), $K_{ab}(P_i)$ and $K_{ab}(P_i, P_j)$, are defined as connected two-coloured sub-graphs of G, respectively having maximum numbers of vertices including P_i (for $K_{ab}(P_i)$), and both of P_i and P_j (for $K_{ab}(P_i, P_j)$), where P_i and P_j are different two vertices of G. ■

If P is coloured with a in $col^4_o(G)$, and is not adjacent to any vertex coloured with b, $K_{ab}(P)$ consists of exclusively one vertex P.

III. BASIC THEOREMS

The following basic theorems are useful for proving the FCT. Detailed proofs will be given in Ohnishi (submitted, 2009a, 2009b).

[Theorem 3.1] For $K_{ab}(P_i, P_j)$ in <u>Definition 2.10</u>, there exists a 2-coloured path $U_{ab}(P_j,P_j)$ as a subgraph of the 2-coloured graph, $K_{ab}(P_i, P_j)$. ■

<u>[Proof]</u> Evident from the definitions of connected graph (<u>Definition 2.5</u>) and vertex 2-coloured graph (<u>Definition 2.8</u>). ■

This theorem means that P_i and P_j are connected by a *2-coloured Jordan curve*, $U_{ab}(P_j,P_j)$.

[Theorem 3.2] Let $T_k(S^2)$ be a complete triangulation of S^2, having k vertices ($k \geq 4$). Then there exists a quadrilateral given by $Q^{2f}_{k,0} = Q^{2f}(C^4_{k,0};e_{13}) \subset T_k$, where $C^4_{k,0} = C^4(e_{12},e_{23},e_{34},e_{41})$, $\langle e_{13} \rangle \subset int\ C^4_{k,0}$, and $e_{ij} = [P_i, P_j]$. Furthermore, $Q^{2f}_{k,0}$ satisfies $val\ P_1 \geq 3$, $val\ P_3 \geq 3$, $val\ P_2 \geq 2$, and $val\ P_4 \geq 2$. ■

<u>[Proof]</u> See Ohnishi [9]. ■

[Lemma 3.2.1] In Theorem 3.2, a set, $\{Q^{2f}_k\}$ is an unavoidable set (See [4] for definition.) of $T_k (S^2)$, and consists of only one element being a quadrilateral. ■

<u>[Proof]</u> Evident from Theorem 3.2.1. ■

[Theorem 3.3] Let $T_k(S^2)$, $C^4_{k,0}=C^4(e_{12},e_{23},e_{34},e_{41})$, and $Q^{2f}_{k,0}=Q^{2f}(C^4_{k,0};e_{13}) \subset T_k$, ($k \geq 4$) be defined as same as in <u>Theorem 3.2</u>,with an additional condition that T_k is 4-colourable. If $col^4_0 (T_k) = col^4(T_k; Q^{2f}_{k,0})$ is a four-cloured complete triangulation graph of T_k coloured with a, b, c, and d, then we can consider, without losing generality, a coloration satisfying $col^4_0 (P_1) = a$, $col^4_0 (P_2) = b$, $col^4_0 (P_3) = c$, and $col^4_0 (P_4) = c$ or d. We find that $col^4_0 (T_k)$ belongs to either one of the following two cases;

case I: There exists $K_{ac}(P_1,P_3)$ ($\subset col^4(T_k;Q^{2f}_0)$).

case II: There does not exist $K_{ac}(P_1,P_3)$ ($\subset col^4 (T_k;Q^{2f}_0)$). ■

<u>[Proof]</u> See Ohnishi [9]. ■

[Definition 3.1] (case I and case II 4-colorations) Let $col^4_{\mathrm{I}}(T_k;Q^{2f}_0)$ and $col^4_{\mathrm{II}}(T_k;Q^{2f}_0)$ respectively denote case I and case II 4-coloured complete triangulation graph described in <u>Theorem 3.3</u>.

IV. VERTEX-REDUCING COMPLETE TRIANGULATION LINEAGE

[Theorem 4.1] Let $T_k(S^2)$, $C^4{}_{k,0} = C^4(e_{12},e_{23},e_{34},e_{41})$, and $Q^{2f}{}_{k,0}=Q^{2f}(C^4{}_0;e_{13}) \subset T_k$, $(k \geq 4)$ be defined as same as in Theorem 3.2. Then $Q^{2f}{}_{k,0}$ belongs to either one of the following three types;

type A: $Q^{2f}{}_{k,0}$ is a complete two-faced quadrilateral. ($val\ P_1 \geq 4$, $val\ P_3 \geq 4$, $val\ P_2 \geq 3$, $val\ P_4 \geq$)

type B: $Q^{2f}{}_{k,0}$ is an incomplete two-faced quadrilateral, in which there exists $e'_{13}=[P_1,P_3] \subseteq Ext\ C^4{}_{k,0}$. ($val\ P_1 \geq 4$, $val\ P_3 \geq 4$, $val\ P_2 \geq 2$, $val\ P_4 \geq 2$)

type C: $Q^{2f}{}_{k,0}$ is an incomplete two-faced quadrilateral, in which there exists $e'_{24}=[P_2,P_4] \subseteq Ext\ C^4{}_{k,0}$. ($val\ P_1 \geq 3$, $val\ P_3 \geq 3$, $val\ P_2 \geq 3$, $val\ P_4 \geq 3$). ▌

[Proof] See Ohnishi [10]. █

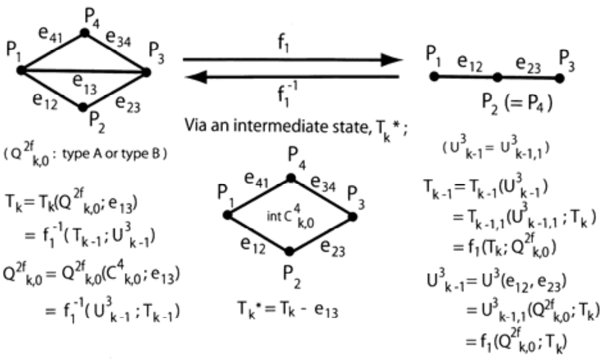

Figure 2. Vertex-reducing operation f_1 and its in verse operation, f_1^{-1}. See Definition 4.1 and Ohni shi [10] for dertails.

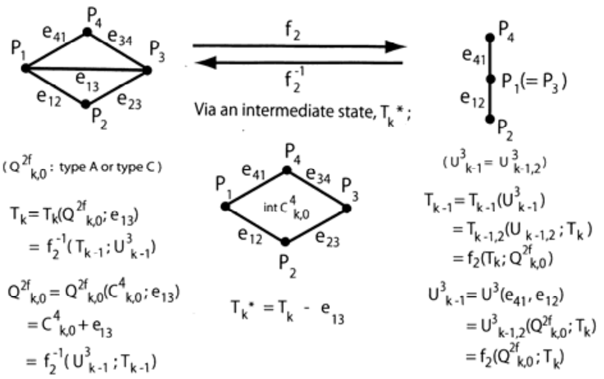

Figure 3. Vertex-reducing operation f_2 and its inverse operation, f_2^{-1}. See Definition 4.1. and Ohnishi [10] for dertails.

[Definition 4.1] (Vertex-reducing operations of quadrilaterals): For type A and type B quadrilaterals ($Q^{2f}{}_{k,0}$) of $T_k(S^2)$, a vertex-reducing operations f_1 which converts $T_k(S^2)=T_k(Q^{2f}{}_{k0};e_{12})$ to $T_{k-1}(S^2)=T_{k-1,1}(U^3{}_{k-1,1};\ T_k)$ are defined as illustrated in Figure 2. Similarly, for type A and type C quadrilaterals, a vertex-reducing operation f_2, which converts $T_k(S^2)= T_k(Q^{2f}{}_{k,0};e_{12})$ to $T_{k-1}(S^2) = T_{k-1,2}(U^3{}_{k-1,2};T_k)$, are defin edas illustrated in Figure 3. Since it is evident that

These operations are reversible via an intermediate state ($T_k{}^* = T_k - e_{13}$) shown in the figures 2 and 3, there exist inverse operations f_1^{-1} and f_2^{-1} converting T_{k-1} to T_k. These relations are expressed by;

$T_{k-1} = f_i(T_k)$, $T_k = f_i^{-1}(T^k)$, $(i = 1,2)$, [#4.1]

or;

$$T_k \overset{f_i/f_i^{-1}}{\longleftrightarrow} T_{k-1} \quad (i = 1,2), \qquad [\#4.2]$$

where $T_{k-1} = T_{k-1,i}(U^3{}_{k-1,i};\ T_k)$, $T_k = T_k(Q^{2f}k;\ e_{13})$, and $U^3{}_{k-1,i} = fi(Q^{2f}{}_k;\ e_{13})$.

More simply, we write [#4.2] as ;

$$T_k \overset{f/f^{-1}}{\longleftrightarrow} T_{k-1} \quad (i = 1,2). \qquad [\#4.3]$$

▌

Thus we have reached the next theorem;

[Theorem 4.2] (See Figure 2 and Figure 3): Let $T_k(S^2) = T_k(Q^{2f}{}_{k,0};e_{13})$, $C^4{}_{k,0} = C^4(e_{12},e_{23},e_{34},e_{41})$, and $Q^{2f}{}_{k,0}=Q^{2f}(C^4{}_0;e_{13}) \subset T_k$, $(k \geq 4)$ be defined as same as in Theorem 3.2. Then we have;

(1) If $Q^{2f}{}_{k,0}$ is type A or type B (in this case, $val\ P_1 \geq 4$, $val\ P_3 \geq 4$): A type 1 vertex-reducing complete triangulation $T_{k-1,1}$ is obtained by f_1;

$T_{k-1,1} = T_{k-1,1}(U^3{}_{k-1,1};T_k) = f_1(T_{k-1};\ Q^{2f}{}_{k,0})$,

where $T_{k-1,1}(U^3{}_{k-1,1};T_k)$ denotes that the *type 1* complete triangulation $T_{k-1,1}$ [= $f_1(T_k;\ Q^{2f}{}_{k,0})$] has a 2-path given by $U^3{}_{k-1,1} = U^3(e_{12},\ e_{23}) = f_1(Q^{2f}{}_{k0};\ T_k)$, meaning that $U^3{}_{k-1,1}$ is generated from $Q^{2f}{}_{k,0}$ by f_1., as shown in Figure 2. In $T_{k-1,1}$, $val\ P_1 \geq 2$, $val\ P_3 \geq 2$, $val\ P_2 \geq 2$ $val\ P_4 \geq 2$.

(2) [Figure 2]: If $Q^{2f}{}_{k,0}$ is type A or type C (in this case, $val\ P_2 \geq 3$, $val\ P_4 \geq 3$): A type 2 vertex-reducing complete triangulation $T_{k-1,1}$ is obtained by f_1;

$T_{k-1,2} = T_{k-1,2}(U^3{}_{k-1,2};T_k) = f_2(T_{k-1};\ Q^{2f}{}_{k,0})$,

where $T_{k-1,2}(U^3{}_{k-1,21};T_k)$ denotes that the *type 2* complete triangulation $T_{k-1,2}$ [= $f_2(T_k;\ Q^{2f}{}_{k,0})$] has a 2-path given by $U^3{}_{k-1,2} = U^3(e_{12},\ e_{23}) = f_2(Q^{2f}{}_{k0};\ T_k)$, meaning that $U^3{}_{k-1,2}$ is generated from $Q^{2f}{}_{k,0}$ by f_2., as shown in Figure 2. In $T_{k-1,2}$, $val\ P_1 \geq 2$, $val\ P_3 \geq 2$, $val\ P_2 \geq 2$ $val\ P_4 \geq 2$.

Thus we finally have;

(i) If $Q^{2f}{}_{k,0}$ is type A (i.e.,complete quadrilateral), then we have

$T_{k-1,1} = f_1(T_k;\ Q^{2f}{}_{k,0}) = T_{k-1,1}(U^3{}_{k-1,1};T_k)$,
$T_{k-1,2} = f_2(T_k;\ Q^{2f}{}_{k,0}) = T_{k-1,2}(U^3{}_{k-1,2};T_k)$.

(ii) If $Q^{2f}{}_{k,0}$ is type B, then we have only

$T_{k-1,1} = f_1(T_k;\ Q^{2f}{}_{k,0}) = T_{k-1,1}(U^3{}_{k-1,1};T_k)$,

(iii) If $Q^{2f}{}_{k,0}$ is type C, then we have only

$T_{k-1,2} = f_2(T_k;\ Q^{2f}{}_{k,0}) = T_{k-1,2}(U^3{}_{k-1,2};T_k)$.

The vertex-reductions, $T_{k-1,i} = f_i(T_k\ ;Q^{2f}{}_{k,0}) = T_{k-1,i}$ $(U^3{}_{k-1,i};T_k\ )$ $(i = 1,2)$ in (i),(ii),(iii) are all reversible by $T_k = T_k\ (Q^{2f}{}_{k,0}\ ;\ e_{13}) = f_i^{-1}(T_{k-1,i};\ U^3{}_{k-1,i})$. *Thus* T_k $(Q^{2f}{}_{k,0}\ ;\ e_{13})$ *can be reconstructed from* $T_{k-1,i}(U^3{}_{k-1,i};$ $T_k\ )$ *by either or both of* f_1^{-1} *and* f_2^{-1}. ▌

[Proof] Evident from the definitions and theorems descibed above. See also Ohnishi [10]. █

[Lemma 4.2.1] For $T_k(S^2) = T_k(Q^{2f}{}_{k,0};e_{13})$ $(k \geq 4)$

in Theorem 4.2, we find

$$T_k \overset{f_i/f_i^{-1}}{\longleftrightarrow} T_{k-1} \qquad [\#4.3a]$$

where $T_{k-1} = T_{k-1,i} = f_i(T_k)$ for at least either one of *i=1 and i=2.*
and

$$T_k \overset{f_{i(k)}/f_{i(k)}^{-1}}{\longleftrightarrow} T_{k-1} \overset{f_{i(k-1)}/f_{i(k-1)}^{-1}}{\longleftrightarrow} T_{k-2} \overset{f_{i(k-2)}/f_{i(k-2)}^{-1}}{\longleftrightarrow} \cdots\cdots \overset{f_{i(k-s+1)}/f_{i(k-s+1)}^{-1}}{\longleftrightarrow} T_{k-s}$$

$$(1 \le s \le k\text{-}3) \qquad [\#4.4]$$

where $i(k\text{-}j) = 1$, *and/or* 2, $j = 1,2,\ldots,s$. ■

[Proof] Easily proven from #4.2 and Theorem 4.2. See Ohnishi [10]. ■

[Lemma 4.2.1] In Lemma 4.2.1, we find

$$T_k \overset{f_{i(k)}/f_{i(k)}^{-1}}{\longleftrightarrow} T_{k-1} \overset{f_{i(k-1)}/f_{i(k-1)}^{-1}}{\longleftrightarrow} T_{k-2} \overset{f_{i(k-2)}/f_{i(k-2)}^{-1}}{\longleftrightarrow} \cdots\cdots \overset{f_{i(4)}/f_{i(4)}^{-1}}{\longleftrightarrow} T_3 \qquad [\#4.5]$$

where $i(k\text{-}j) = 1$, *and/or* 2, $j = 1,2,\ldots,k\text{-}4$, *and*

$T_{k-j-1}= f_{i(k-j)}(T_{k-j};Q^{2f}_{k-j,0})$, $T_{k-j} = f_{i(k-j)}^{-1}(T_{k-j-1};\ U^3_{k-j-1})$. ■

[Proof] Proven by letting $s = k\text{-}3$, in [#4.4]. ■

V. TOWARDS FINAL PROOF OF THE FOUR COLOUR THEOREM

[Theorem 5.1] A necessary and sufficient conditio n for that an arbitrary complete triangulation $T_k(S^2)$ (with k vertices, $k \ge 4$) is vertex 4-colourable is as below;

Under the assumption that T_{k-j} is vertex 4-colou-rable, there exists a 4-coloured graph, $col^4{}_0(T_{k-j-1})$, $(j= 1,2,\ldots,j\text{-}3)$, which can be derived from a hypo-thetical 4-coloured graph of T_{k-j}, given by $col^4{}_0(T_{k-j})$, where $T_{k-j-1}= f(T_{k-j};\ Q^{2f}_{k-j,0})$, $f = f_1$ or f_2 (defined by Definition 4.1) and $Q^{2f}_{k-j,0} = Q^{2f}(C^4_{k-j,0}\ ;e^{(k-j)}_{13})$. Here

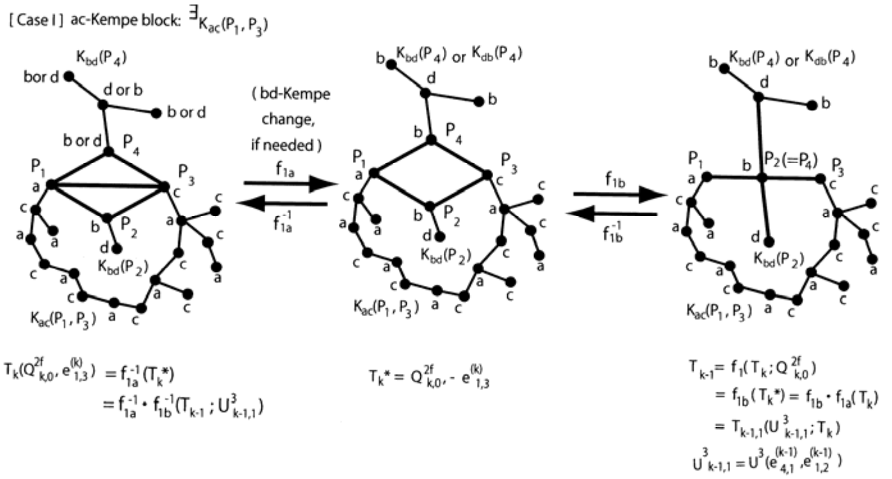

Figure 4. Vertex-reduction of a 4-coloured complete triangulation graph, $col^4{}_0(T_k\ ;\ Q^{2f}_{k,0})$ in case I. See text and Ohnishi [10] for details.

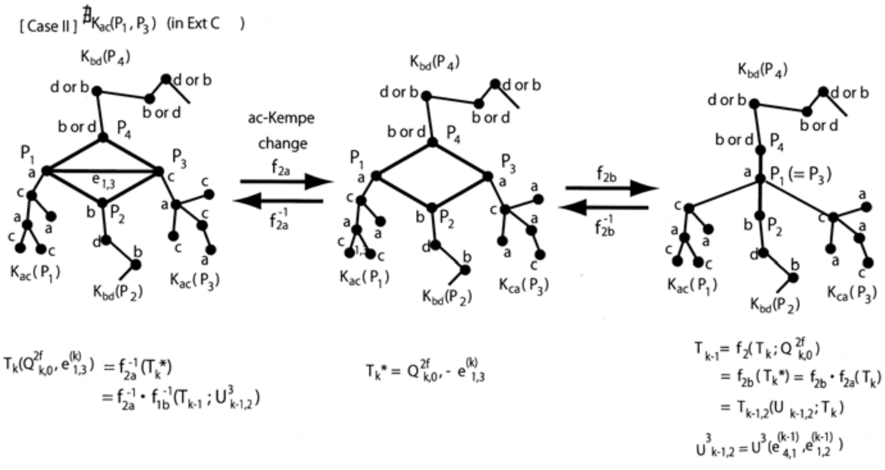

Figure 4. Vertex-reduction of a 4-coloured complete triangulation graph, $col^4{}_0(T_k\ ;\ Q^{2f}_{k,0})$ in case II. See text and Ohnishi [10] for details.

$C^4_{k-j,0}$ is a 4-cycle of T_{k-j}, and $Q^{2f}(C^4_{k-j,0}\ ;e^{(k-j)}_{13})$ is a quadrilateral consisting of $C^4_{k-j,0} = C^4(e^{(k-j)}_{12}, e^{(k-j)}_{23}, e^{(k-j)}_{34}, e^{(k-j)}_{41})$, and $e^{(k-j)}_{i1,i2} = [P^{(k-j)}_{i1}, P^{(k-j)}_{i2}]$. ▌

Proof For j = 0, we need to proof

$$col^4_0(T_k) \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_{k-1}). \quad [\#4.6]$$

From Theorem 3.3 and Definition 3.1, we find
$col^4_0(T_k) = col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})$ *(for case I)* *[Eq.*
or $col^4_{\mathrm{II}}(T_k;Q^{2f}_{k,0})$. *(for case II ,*
and we can consider that $col^4_0\ (P^{(k)}_1)=a$, $col^4_0(P^{(k)}_2) = b$, $col^4_0\ (P^{(k)}_3) = c$, *and* $col^4_0\ (P^{(k)}_4) = b$ *or* d, without losing generality.

[I] (See Figure 4) In *case I,* $^\exists K_{ac}(P^{(k)}_1,\ P^{(k)}_3) \subseteq Ext\ C^4_{k-j,0}$, and then there exists a *ac*-coloured Jordan curve path *connecting* $P^{(k)}_1$ to $P^{(k)}_3$. Accordingly, $K_{bd}(P^{(k)}_4)$ can be changed into $K_{db}(P^{(k)}_4)$, if $col^4_0\ (P^{(k)}_4) = d$. *Thus we have a 4-colouring of* T_k^* (= $T_k - e^{(k)}_{13}$) *with both* $P^{(k)}_2$ and $P^{(k)}_4$ with b. Let f_{1a} deno te the operation converting $col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})$ *to this 4-colouring,* $col^4_{\mathrm{I},1\alpha}(T_k^*;C^4_{k,0}\ )$, then we have $col^4_{\mathrm{I},1\alpha}(T_k^*;C^4_{k,0}\ ) = f_{1a}[col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})]$. *It is evident that t his operation is reversible, and therefore,* $col^4_{\mathrm{I}}(T_k\ ;\ Q^{2f}_{k,0}) = f_{1a}^{-1}[col^4_{\mathrm{I},1}(T_k^*;C^4_{k,0}\ )]$.
Since $col^4_0(P^{(k)}_2) = col^4_0(P^{(k)}_4) = b$, $col^4_{\mathrm{I},1}(T_k^*;C^4_{k,0})$ c an be further modified to the four coloured graph, $col^4_0(T_{k-1};U^3_{k-1,1})$, where $U^3_{k-1,1} = U^3(e^{(k-1)}_{4,1},\ e^{(k-1)}_{12})$.
Let f_{1b} denote the operation converting $col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})$ *to* $col^4_0(T_{k-1};U^3_{k-1,1})$, as shown in Figure 4. Then we have $col^4_0(T_{k-1};U^3_{k-1,1}) = f_{1b}[col^4_{\mathrm{I},1\alpha}(T_k^*;C^4_{k,0}\ )] = f_{1b}[f_{1a}[col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})]\ ] = f_{1b} \circ f_{1a}[col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})] = f_1[col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})]$, $(f_1 \equiv f_{1b} \circ f_{1a})$. Note that every vertex in $T_{k-1,1}$ shows *valency* ≥ 2, since $Q^{2f}_{k,0}$ is ty pe A or type B (from Theorem 4.1).

It is also evident that f_{1b} is also reversible, and we have $col^4_{\mathrm{I},1\alpha}(T_k^*;C^4_{k,0}\ ) = f_{1b}^{-1}[col^4_0(T_{k-1};U^3_{k-1,1})]$, and therefore, $col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0}) = f_{1a}^{-1}[col^4_{\mathrm{I},1\alpha}(T_k^*;C^4_{k,0}\ )] = f_{1a}^{-1}[f_{1b}^{-1}[col^4_0(T_{k-1};U^3_{k-1,1})]] = f_{1a}^{-1} \circ f_{1b}^{-1}[col^4_0(T_{k-1};U^3_{k-1,1})] = f_1^{-1}[col^4_0(T_{k-1};U^3_{k-1,1})]$. Thus for *case I, where* $col^4_0\ (T_k) = col^4_{\mathrm{I}}(T_k;Q^{2f}_{k,0})$, we find

$$col^4_0\ (T_k) \overset{f_1/f_1^{-1}}{\longleftrightarrow} col^4_0(T_{k-1}), \quad [\#4.7]$$

although $col^4_0(T_{k-1};U^3_{k-1})$ is defined as derived from $col^4_0\ (T_k;\ Q^{2f}_{k,0})$. In other words, for a given $T_k(Q^{2f}_{k,0};\ e^{(k)}_{1,3})$ really exists as proven in Theorem 4.2., but the existence of $col^4_0\ (T_k)$ is an assumption, an d therefore, the relation [#4.7] could have some me aning only if $col^4_0(T_{k-1})$ could exist.

[II] (See Figure 5) In *case II,* there does not exista ny *ac*-Kempe block satisfying $K_{ac}(P^{(k)}_1,P^{(k)}_{33}) \subseteq Ext\ C^4_{k-j,0}$, which means no existence of any *ac*-coloured Jordan curve path *connecting* $P^{(k)}_1$ to $P^{(k)}_3$. Accordingly, in T_k^*, a conversion of $K_{ac}(P_3)$ to $K_{ca}(P_3)$ *generates* P_3 *and* P_1 *to be coloured with the s ame colour, a.* Thus, as similarly as in case I, we find, for *case II, where* $col^4_0\ (T_k) = col^4_{\mathrm{II}}(T_k;Q^{2f}_{k,0})$, we find

$$col^4_0\ (T_k) \overset{f_2/f_2^{-1}}{\longleftrightarrow} col^4_0(T_{k-1}), \quad [\#4.8]$$

although $col^4_0(T_{k-1})$ is defined as deried from $col^4_{\mathrm{II}}(T_k;Q^{2f}_{k,0})$ (See Figure 5).

Since $col^4_0\ (T_k;Q^{2f}_{k,0}) = col^4_{\mathrm{I}}\ (T_k;Q^{2f}_{k,0})$ *and/or* $col^4_{\mathrm{II}}(T_k;Q^{2f}_{k,0})$, *#4.7 and #4.8 means #4.6 is true, if* $col^4_0(T_{k-1})$ could exist.

By converting *k into k-j,* #4.8 proves the Theorem 5.1,

$$col^4_0\ (T_{k-j}) \overset{f_2/f_2^{-1}}{\longleftrightarrow} col^4_0(T_{k-j1}). \quad [\#4.8a]$$ ▌

However, it is important to note that we do not know whether or not $col^4_0\ (T_{k-1})$ could exist in #4.8.
If $col^4_0\ (T_{k-1})$ exists, then we similarly find

$$col^4_0(T_{k-1}) \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_{k-2}). \quad [\#4.6a]$$

By repeating similar considerations, we easily find an important Theorem;

[Theorem 5.2] If T_k(S2) mentioned in Theorem 4.1 is fourcolourable, we find the followings, #4.9 ~ #4.10a;

$$col^4_0(T_{k-j}) \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_{k-j-1}), \quad [\#4.9]$$

(j = 0,1,2, ..., k-s-1), (s ≤ k-4),
or,

$$col^4_0(T_k) \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_{k-1}) \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_{k-2}) \overset{f/f^{-1}}{\longleftrightarrow} \ldots \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_{s+1}) \overset{f/f^{-1}}{\longleftrightarrow} col^4_0(T_s), \quad [\#4.9a].$$

where

$$col^4_0(T_s) = f[col^4_0(T_{s+1})] = f^2[col^4_0(T_{s+1})] = \ldots = f^{k-s}[col^4_0(T_k)], \quad [\#4.10]$$

$$col^4_0(T_k) = f^{-1}[col^4_0(T_{k-1})] = (f^{-1})^2[col^4_0(T_{k-1})] = \ldots = (f^{-1})^{k-s}[col^4_0(T_s)]. \quad [\#4.10a]$$

In #4.9/#4.9a, we do not know $col^4_0(T_s)$, although we can obtain, from Lemma 4.2.1, a *s*-vertex com- plete triangulation graph, T_s, deduced from T_k.

[Lemma 5.2.1] *Based on Theorem 5.2, it follows;*

$$col^4_0(T_k) \overset{f^{k-s}/(f^{-1})^{k-s}}{\longleftrightarrow} col^4_0(T_s), \quad (s \leq k-4), \quad [\#4.10]$$

or, $col^4_0(T_s) = f^{k-s}[col^4_0(T_k)]$, $col^4_0(T_k) = (f^{-1})^{k-s}[col^4_0(T_s)]$. ▌

Proof Evoident from above descriptions. ▌

Thus we have reached the Final Theorem;

[Theorem 5.3] (The Four Colour Theorem): Eve ry complete triangulation of S^2, $T_k(S^2)$, is vertex fou r colourable *(k ≥ 3)*. ▌

[Proof] For k *(k ≥ 4)*, let s = 3 in Lemma 5.2.1, we find;

$$col^4_0(T_k) \overset{f^{k-s}/(f^{-1})^{k-3}}{\longleftrightarrow} col^4_0(T_3), \quad [\#4.11]$$

or, $col^4_0(T_3) = f^{k-3}[col^4_0(T_k)]$, [#4.11a]
$col^4_0(T_k) = (f^{-1})^{k-3}[col^4_0(T_3)]$. [#4.11b]

#4.11 ~ #4.11b is a special case of Lemma 5.2.1, where $col^4_0(T_3)$ really exists as one of the possible *4x3x2 (=24) 4*-colourings of a triangle, $T^3(S^2)$, by appropriately naming vertices.

Thus we can safely conclude that;

(1) The necessary and sufficient condition for the existence of a 4-coloured complete triangulation, $col^4{}_0(T_k)$, is the existence of $col^4{}_0(T_3)$, satisfying $col^4{}_0(T_3) = f^{k-3}[col^4{}_0(T_k)]$.

(2) "The $col^4{}_0(T_3)$" really exists.

From (1) and (2), Theorem 5.3 is proved. ■

VI. CONCLUTION

The essential portion of the proof of the FCT is briefly presented. The entire, complete proof will be published elsewhere. It is most important that the essence of the enormous complexity of the FCT is beautifully found in the series of the necessary a nd sufficient conditions given in #4.9a (for $s=3$).

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