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Harmonic oscillations in Lotka-Volterra dynamic systems: A new approach from a matrix operator equation system

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Abstract: **A 2x2 matrix operator $F=(a_{ij})$, satisfying a harmonic oscillation-type operator equation, $F^2x=-\omega^2x$, (where $x={}^t(x_1(t), x_2(t))$), was obtained by letting $a_{22}= -a_{11}$ and $a_{12}a_{21} = -(\omega^2+a_{11}^2)$. For a_{ij} = const. or $= a_{ij}(t)$, and for some cases of $a_{ij} = a_{ij}(x_1, x_2, t)$, general solutions of $F^2x= -\omega^2x$ were obtained, discussed, and applied to some cases. Mathematical relationship between $F^2x = -\omega^2x$ and $d^2x/dt^2 = -\omega^2x$, where $dx/dt = Fx$, was also analyzed. An application was made to Lotka-Voltera dynamic system, where $a_{11} = r_1-\alpha x_1-\beta x_2$, $a_{12}= -ax_1$, $a_{21} = a'x_{21}$, $a_{22} = -a_{11}$. Finally, this type of Lotka-Volterra operator equation was found to have a trajectory given by an ellipse (for $C > 0$), $(X_1 - X_{10})^2/(C/\kappa_1) + (X_2 - X_{20})^2/(C/\kappa_2) =1$, where $X_i = \pm[\{(\alpha^2-\beta^2+\sigma_1\sqrt{D_0})/(2h)\}x_1+x_2]/\sqrt{\kappa_i\sqrt{D_0}/h^2}$, ($i$=1,2), in which $(\sigma_1,\sigma_2)= (1,-1)$, $h = \alpha\beta - aa'$, $D_0=(\alpha^2+\beta^2)^2 + 4h^2$, $\kappa_i =(\alpha^2+\beta^2+D_0^{1/2})/2$, $C= (r_1^2/4)\{(\alpha v_{11} +\beta v_{21})^2/\kappa_1 + (\alpha v_{12}+\beta v_{22})^2/\kappa_2\} - (r_1^2 + \omega^2)$. Schroedinger equation was also discussed from this aspect of generalized harmonic oscillation.**

***Keywords*:** harmonic oscillator system, operaror equation, Lotka-Volterra equation.

1. Introduction

Lotka-Volterra differential equation system has long been analyzed from various aspects in many different and interdisciplinary scientific fields (Reijenge, 2002; Hofbauer & Sigmund, 1998). A well-known typical differential equation system of Lotka-Volterra type is given by

$$(d/dt)x \equiv \begin{pmatrix} dx_1/dt \\ dx_2/dt \end{pmatrix} = \begin{pmatrix} rx_1 - ax_1x_2 \\ a'x_1x_2 - r'x_2 \end{pmatrix} \quad \text{[Eq.1]}$$

where $x = {}^t(x_1, x_2)$, and

$$r = r_1 - \alpha_1x_1 - \beta_1x_2, \quad \text{[Eq.1b]}$$

$$r' = r_2 - \alpha_2x_1 - \beta_2x_2,. \quad \text{[Eq.1c]}$$

In Eqs.1-1b, x_1 and x_2 denote the number of individuals of preys and predators, respectively, and, r and $-r'$ denote the rate of the increase in the number of prey and predator individuals, respectively, both of which depend on t. For i =1 and 2, $r_i, a, a', \alpha_i, \beta_i$ are constants.

In order to analyze this system in relation to harmonic oscillation system, let us consider a 2 x 2 real matrix operator F, given by

$$F = (a_{ij})_{2,2} \stackrel{d}{=} {}^t(\ (a_{11},a_{12}), (a_{21},a_{22})\ ) = \begin{pmatrix} a_{11}, a_{12} \\ a_{21}, a_{22} \end{pmatrix} \quad \text{[Eq.2]}$$

and

$$Q = (q_{ij})_{2,2} = F^2, \quad \text{[Eq.3]}$$

satisfying

$$Qx = F^2x = -\omega^2x, \ (\omega^2 > 0) \quad \text{[Eq.4]}$$

in which x denotes a real vector, $x= {}^t(x_1(t),x_2(t))$, where ^{t} denotes transpose, and a_{ij} are either constants (in simpler cases) or functions of $x_1(t)$, $x_2(t)$, and/or t (i.e., $a_{ij}=a_{ij}(x_1,x_2,t)$, in more general cases). Eq.4 means that $-\omega^2$ (< 0) is an eigenvalue or eigenfunction of Q.

From Eqs.2-4, eigenequation of Q is given by

$$g(\kappa) = g(-\omega^2) = \begin{vmatrix} (a_{11}{}^2 + a_{12}a_{21}) + \omega^2, & a_{12}(a_{11} + a_{22}) \\ a_{21}(a_{11} + a_{22}), & (a_{22}{}^2 + a_{12}a_{21}) + \omega^2 \end{vmatrix}$$
$$= (a_{11}+a_{22})^2\omega^2 + \{\omega^2 - (a_{11}a_{22} - a_{12}a_{21})\}^2 = 0, \quad \text{[Eq.5]}$$

where κ $(=-\omega^2)$ is an eigenvalue of Q $(= F^2)$. Accordingly, the necessary and sufficient condition for the existence of non-zero solution x of the operator equation, Eq.4, is given by Eq.5.

Since $\omega^2 > 0$, we find, from Eq.5 and Eq.6, that

$$a_{11} + a_{22} = 0, \quad \text{[Eq.6]}$$

and

$$\omega^2 - (a_{11}a_{22} - a_{12}a_{21}) = 0. \quad \text{[Eq.7]}$$

Eqs.6-7 bring about

$$\omega^2 + (a_{11}{}^2 + a_{12}a_{21}) = 0. \quad \text{[Eq.8]}$$

We easily find

$$a_{12} \neq 0, a_{21} \neq 0, \quad \text{[Eq.9]}$$

because $Q = F^2 = {}^t(\ (a_{11}{}^2,0), (0,a_{22}{}^2)\ )$ does not satisfy Eq.4.

Notice that Eqs.5-9 can be obtained not only for the case of a_{ij} = const. or $= a_{ij}(t)$ (*i.e.,* functions of t), but also for the general case of $a_{ij} = a_{ij}(x_1,x_2,t\ )$ or for other cases where a_{ij} are arbitrarily selected functions. This is because the processes from Eq.2 to Eq.4 do not include any operation of "differentiation" (such as d/dt), and consist exclusively of multiplications by a matrix F.

From Eqs.6-9, we finally obtain

$$F = \begin{pmatrix} a_{11}, & a_{12} \\ -(a_{11}{}^2 + \omega^2)/a_{12}, & -a_{11} \end{pmatrix}. \qquad \text{[Eq.10]}$$

Notice that Eq.4 is completely satisfied by Eq.10, even if ω is any real function of t and/or other variables such as $\omega = \omega(t)$ or $\omega = \omega(x_1(t), x_2(t), t, \ldots)$, the latter being an explicit function of x_1 and x_2.

For special cases where $a_{11} = 0$, and where $a_{11} = 0$, $a_{12} = -\omega$, Eq.10 means

$$F = {}^t((0,a_{12}), (-\omega^2/a_{12},0)), \qquad \text{[Eq.11]}$$

and

$$F = {}^t((0,-\omega),(\omega,0)) = \omega\, {}^t((0,-1),(1,0)), \qquad \text{[Eq.11a]}$$

respectively. Noticing that Eq.12a gives

$$F = {}^t((0,-1),(1,0)), (=I, \text{ as below.}), \qquad \text{[Eq.11b]}$$

if $\omega = 1$, F in Eq.11b can be considered to be a further generalization of $I = {}^t((0,-1),(1,0))$ (satisfying $I^2 = -E$), which is often used as a matrix operator expression of the imaginary unit $\boldsymbol{i}$ ($=\sqrt{-1}$). Accordingly, an alternative definition for matrix operator expression of $(-1)^{1/2}$,

$$I = \begin{pmatrix} a_{11}, & a_{12} \\ -(a_{11}{}^2 + 1)/a_{12}, & -a_{11} \end{pmatrix}, \qquad \text{[Eq.12]}$$

can satisfy $I^2 = -E$, and would be expected to have some useful characteristics not possessed by the conventional matrix operator expression given in Eq.11b, since Eq.11b is considered to be a special case of Eq.12, which is again a special case of F defined byEq.11.

2. Mathematical relations of a matrix equation, $F^2x = -\omega^2 x$, to a differential equation, $d^2x/dt^2 = -\omega^2 x$ (where $dx/dt = Fx$)

The matrix operator equation, $F^2x = -\omega^2 x$ (Eq.4), is a kind of generalization of

$$D^2u(t) \overset{d}{=} (d^2/dt^2)u(t) = -\omega^2 u(t),$$

which gives a harmonic oscillation of a scalar function $u(t)$, with an angular frequency ω, where D is a differential operator, $D = d/dt$.

In this section, by letting $x = {}^t(x_1(t), x_2(t))$ and $F = (a_{ij})_{2,2}$ be a real number vector and a real number matrix, we shall compare the relation between a harmonic oscillator-like matrix operator equation given by

$$F^2x = -\omega^2 x$$

and a simultaneous differential equation,

$$Dx = Fx, \qquad \text{[Eq.13a]}$$

satisfying a harmonic oscillator equation,

$$D^2x \overset{d}{=} {}^t(D^2x_1, D^2x_2) = -\omega^2 x. \qquad \text{[Eq13b]}$$

Here we consider a generalized case of Eq.4, where a_{ij} are either constants or functions of $x_1(t)$, *and* $x_2(t)$, and t, *i.e.*,

$$a_{ij} = a_{ij}(x_1, x_2, t). \qquad \text{[Eq.14]}$$

Then we have,

$$Fx = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}, \qquad \text{[Eq.14a]}$$

$$F^2 = \begin{pmatrix} (a_{11}{}^2 + a_{12}a_{21})x_1 + a_{12}(a_{11} + a_{22})x_2 \\ a_{21}(a_{11} + a_{22})x_1 + (a_{22}{}^2 + a_{12}a_{21})x_2 \end{pmatrix}. \qquad \text{[Eq.14b]}$$

A simultaneous differential equation, given by

$$Dx \overset{d}{=} {}^t(Dx_1, Dx_2) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}, \qquad \text{[Eq.15]}$$

is now considered for comparison with Eqs.14a,b. From Eq.14a and Eq.15, we first find

$$Dx = Fx.$$

On the other hand,

$$D^2x = D(Dx) = D\,{}^t(Dx_1, Dx_2)$$
$$= D\begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix}$$
$$= \begin{pmatrix} (a_{11}Dx_1 + a_{12}Dx_2) + ((Da_{11})x_1 + (Da_{12})x_2) \\ (a_{21}Dx_1 + a_{22}Dx_2) + ((Da_{21})x_1 + (Da_{22})x_2) \end{pmatrix}$$
$$= \begin{pmatrix} a_{11}, a_{12} \\ a_{21}, a_{22} \end{pmatrix}\begin{pmatrix} Dx_1 \\ Dx_2 \end{pmatrix} + \begin{pmatrix} (Da_{11})x_1 + (Da_{12})x_2 \\ (Da_{21})x_1 + (Da_{22})x_2 \end{pmatrix}. \qquad \text{[Eq.16]}$$
$$= F(Dx) + \begin{pmatrix} (Da_{11})x_1 + (Da_{12})x_2 \\ (Da_{21})x_1 + (Da_{22})x_2 \end{pmatrix}.$$

Since $F(Dx) = F(Fx) = F^2x$, then

$$D^2x = F^2x + \begin{pmatrix} (Da_{11})x_1 + (Da_{12})x_2 \\ (Da_{21})x_1 + (Da_{22})x_2 \end{pmatrix} \qquad \text{[Eq.17]}$$

$$= F^2x + \begin{pmatrix} Da_{11}, Da_{12} \\ Da_{21}, Da_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \text{[Eq.18]}$$

$$= F^2x + (DF)x. \qquad \text{[Eq.18a]}$$

From Eqs.17-18, we finally have

$$Da_{11}/Da_{12} = Da_{21}/Da_{22} (= -x_2/x_1),$$

or

$$Da_{11}Da_{22} = Da_{12}Da_{21}, \qquad \text{[Eq.19]}$$

as the necessary and sufficient condition for obtaining

$$D^2x = F^2x. \qquad \text{[Eq.20]}$$

Under the condition of Eq.7, we find

$$Da_{22} = -Da_{11}, \qquad \text{[Eq.21a]}$$

and therefore Eq.19 leads to

$Da_{12}Da_{21} = -(Da_{11})^2$. [Eq.21b]

Accordingly, we now get the following theorem;

[Theorem 1.]

Let x and F be a real vector and a real matrix operator, respectively given by $x = {}^t(x_1(t), x_2(t))$ and $F=(a_{ij})_{2,2}$, where $a_{ij} = a_{ij}(x_1,x_2,t)$. If $F^2x = -\omega^2x$, $Dx = Fx$, and $Da_{12}Da_{21} = -(Da_{11})^2$, (where $D=d/dt$), then we get a harmonic oscillator, $D^2x = -\omega^2x$, which further gives

$D^2x\ (=d^2x/dt^2) = F^2x = -\omega^2x$. [Eq.22]

[Proof] From $F^2x = -\omega^2x$, we find $a_{22}=-a_{11}$ and $a_{21}=-(a_{11}{}^2+\omega^2)$, which further gives Eq.22 under the condition of $Dx = Fx$ and $Da_{12}Da_{21}= - (Da_{11})^2$, as described above.

From Eq.21b, it is evident that

$Da_{11}= Da_{12}=0$ [Eq.21c]

(which means that a_{11} and a_{12} are constants.) is sufficient for obtaining Eq.22, if we have $F^2x = -\omega^2x$ and $dx/dt =Fx$. It is also quite evident that

$Da_{ij} = 0, (i,j = 1,2)$ [Eq.21d]

(i.e., a_{ij} : constants) give harmonic oscillation of $x(t)$ given by Eq.22, under the similar conditions.

In case of Lotka-Volterra system in Eqs.1-1b, where $a_{12}=ax_1$, $a_{21}= -a'x_2$, $a_{11}= r_1-\alpha x_1-\beta x_2$, $a_{22}=-a_{11}$, Eq.21b is written as

$(\alpha Dx_1+\beta Dx_2)^2= aa'Dx_1Dx_2$, [Eq.23]

if $Dr_1 =0$ (*i.e.*, r_1: constant). Eq.23 is rewritten as

$\alpha^2(Dx_1/Dx_2)+\beta^2(Dx_2/Dx_1)=-(aa'-2\alpha\beta)$, [Eq.23a]

or as

$\{(\alpha Dx_1+\beta Dx_2)/(\alpha+\beta)\}^2/(Dx_1Dx_2)=aa'/(\alpha+\beta)^2$. [Eq.23b]

3. Towards finding general solutions of $F^2x = -\omega^2x$

In this section, we will attempt to get a general solution of the operator equation given by Eq.4.

Let G_1 *and* G_2 be 2 X 2 matrix operators defined by

$G_1 = e^{tF}$, $G_2 = e^{-tF}$, [Eq.24]

where F is given by Eqs.2-4, satisfying Eq.11. Then we find, by noticing $F^2 = -\omega^2E$, that

$$G_1 = e^{tF} = \sum_{k=0}^{\infty} c_k (tF)^n$$

$$= \sum_{m=0}^{\infty} \{c_{2m}(tF)^{2m} + c_{2m+1}(tF)^{2m+1}\}$$

$$= \sum_{m=0}^{\infty} \{c_{2m}t^{2m}(F^2)^m + c_{2m+1}t^{2m+1}(F^2)^m F\}$$

$$= \sum_{m=0}^{\infty} \{c_{2m}t^{2m}(-\omega^2E)^m + c_{2m+1}t^{2m+1}(-\omega^2E)^m F\}$$

$$= \{\sum_{m=0}^{\infty} c_{2m}(\omega t)^{2m}(-1)^m\}E$$

$$+ \omega^{-1}\{\sum_{m=0}^{\infty} c_{2m+1}(\omega t)^{2m+1}(-1)^m\}F$$

$$= (\cos\omega t)E + (\omega^{-1}\sin\omega t)F ,$$

where $c_k = 1/k!$.

By similar consideration on G_2, we finally obtain

$G_1 = e^{tF} = (\cos\omega t)E + (\omega^{-1}\sin\omega t)F$, [Eq.25.1a]

$$= \begin{pmatrix} \cos\omega t + a_{11}\omega^{-1}\sin\omega t, & a_{12}\omega^{-1}\sin\omega t \\ a_{21}\omega^{-1}\sin\omega t, & \cos\omega t + a_{22}\omega^{-1}\sin\omega t \end{pmatrix},$$ [Eq.25.1b]

$G_2 = e^{-tF} = (\cos\omega t)E - (\omega^{-1}\sin\omega t)F$ [Eq.25.2a]

.

$$= \begin{pmatrix} \cos\omega t - a_{11}\omega^{-1}\sin\omega t, & -a_{12}\omega^{-1}\sin\omega t \\ -a_{21}\omega^{-1}\sin\omega t, & \cos\omega t - a_{22}\omega^{-1}\sin\omega t \end{pmatrix},$$ [Eq.25.2b]

where $a_{22} = -a_{11}$, and $a_{12} = -(a_{11}{}^2+\omega^2)/a_{12}$.

By letting $C_k= {}^t(C_{k1},C_{k2})$, where C_{ki}, ($k,i = 1,2$) are real constants selectable arbitrarily, it is easily found that the two solutions (Eq.26) written below satisfy Eq.4 and are lineally independent solutions (singular solutions) of the operator equation Eq.4, if a_{ij} = const. or = $a_{ij}(t)$, which are not explicit functions of x_1 and x_2;

$x(k)= {}^t(x_{k1},x_{k2}) = G_kC_k$

$= e^{\sigma(k)tF}C_k$

$$= \begin{pmatrix} C_{k1}\cos\omega t + (C_{k1}a_{11} + C_{k2}a_{12})\omega^{-1}\sin\omega t \\ C_{k2}\cos\omega t + (C_{k1}a_{21} + C_{k2}a_{22})\omega^{-1}\sin\omega t \end{pmatrix},$$

(k=1,2, and $\sigma(1)=1,\sigma(2)=-1$, $a_{22}=-a_{11}$). [Eq.26]

Accordingly, general solution of Eq.4 is given by

$x = {}^t(x_1, x_2)$

$= A_1x(1)+A_2x(2)$ [Eq.27a]

$= A_1e^{tF}C_1 + A_2e^{-tF}C_2$ [Eq.27b]

$$= A_1\begin{pmatrix} C_{11}\cos\omega t + (C_{11}a_{11} + C_{12}a_{12})\omega^{-1}\sin\omega t \\ C_{12}\cos\omega t + (C_{11}a_{21} + C_{12}a_{22})\omega^{-1}\sin\omega t \end{pmatrix}$$

$$+ A_2\begin{pmatrix} C_{21}\cos\omega t + (C_{21}a_{11} + C_{22}a_{12})\omega^{-1}\sin\omega t \\ C_{22}\cos\omega t + (C_{21}a_{21} + C_{22}a_{22})\omega^{-1}\sin\omega t \end{pmatrix}$$ [Eq.27c]

$$= \begin{pmatrix} b_{11}\cos\omega t + (b_{12}a_{11} + b_{22}a_{12})\omega^{-1}\sin\omega t \\ b_{21}\cos\omega t + (b_{12}a_{21} + b_{22}a_{22})\omega^{-1}\sin\omega t \end{pmatrix}$$, [Eq.28]

where $a_{22} = -a_{11}$, and $a_{12} = -(a_{11}{}^2+\omega^2)/a_{12}$,

and further, b_{ij} are given by

$b_{11}= A_1C_{11}+ A_2C_{21}$, $b_{12}= A_1C_{11}- A_2C_{21}$,
$b_{21}= A_1C_{12}+ A_2C_{22}$, $b_{22}= A_1C_{12}- A_2C_{22}$. [Eq.28a]

The four equations in Eq.28a mean that the four constants b_{ij} can be arbitrarily selected since the four constants C_{ij} and the two consonants A_1, A_2 are arbitrarily selectable ones.

Eq.28 elucidates a general solution of the operator equation, Eq.4, if a_{ij} in Eq.28 are constants or functions (for example, of t) other than explicit functions of $x_1(t)$ and/or $x_2(t)$. In those cases where some of a_{ij} are explicit function(s) of $x_1(t)$ and/or $x_2(t)$, Eq.28 gives a simultaneous equation of $x_1(t)$ and $x_2(t)$, on which solvability concerning $x_1(t)$ and $x_2(t)$ depends. In those cases where some of a_{ij} are linear combination(s) of $x_1(t)$ and/or $x_2(t)$, Eq.28 might be a solvable simultaneous equation from which general solutions could be deduced, as will be shown later in Lotka-Volterra systems. If Eq.21b or Eq.23 is satisfied, the general solution Eq.28 and that of the simultaneous differential equation Eq.15 are identical.

Eq.28 is rewritten as

$$x_1 = B_{11}\cos\omega t + B_{12}\sin\omega t, \qquad \text{[Eq.29a]}$$

$$x_2 = B_{21}\cos\omega t + B_{22}\sin\omega t, \qquad \text{[Eq.29b]}$$

where

$$B_{11} = b_{11}, \quad B_{12} = (b_{12}a_{11} + b_{22}a_{12})/\omega,$$

$$B_{21} = b_{21}, \quad B_{22} = (b_{12}a_{21} + b_{22}a_{22})/\omega.$$

and where $a_{22} = -a_{11}$, and $a_{12} = -(a_{11}{}^2 + \omega^2)/a_{12}$.

Thus we have

$$\cos\omega t = (B_{22}x_1 - B_{12}x_2)/(B_{11}B_{22} - B_{12}B_{21}),$$

$$\sin\omega t = (-B_{21}x_1 + B_{11}x_2)/(B_{11}B_{22} - B_{12}B_{21}),$$

and therefore it reveals that

$$(B_{22}x_1 - B_{12}x_2)^2 + (-B_{21}x_1 + B_{11}x_2)^2 = (B_{11}B_{22} - B_{12}B_{21})^2, \qquad \text{[Eq.30a]}$$

which further brings about a quadratic equation:

$$(B_{22}{}^2 + B_{21}{}^2)\,x_1{}^2 - 2(B_{12}B_{22} + B_{21}B_{11})\,x_1x_2 + (B_{11}{}^2 + B_{12}{}^2)\,x_2{}^2 - (B_{11}B_{22} - B_{12}B_{21})^2 = 0. \qquad \text{[Eq.30b]}$$

Eq.30b gives a trajectory of Eq.4 ($F^2x = -\omega^2x$; $\omega^2 > 0$) in (x_1, x_2)-plane, which is a conic curve, if B_{ij} are constants.

From Theorem 1, Eq.30b generates a harmonic oscillation given by $D^2x = F^2x = -\omega^2x$ (in Eq.21), if $Dx = Fx$, and $(\alpha Dx_1 + \beta Dx_2)^2 = aa' Dx_1Dx_2$.

4. Lotka-Volterra harmonic oscillations: Mathematical consideration

Lotka-Volterra differential equation in Eq.1 is comparable to the following matrix operator equation.

$$Fx = \begin{pmatrix} a_{11}, a_{12} \\ a_{21}, a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathrm{r}_1 - \alpha_1 \mathrm{x}_1 - \beta_1 x_2, & -ax_1 \\ a'x_2, & -(\mathrm{r}_2 - \alpha_2 \mathrm{x}_1 - \beta_2 x_2) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} (\mathrm{r}_1 - \alpha_1 \mathrm{x}_1 - \beta_1 x_2)x_1 - ax_1x_2, \\ a'\mathrm{x}_1 x_2 - (\mathrm{r}_2 - \alpha_2 \mathrm{x}_1 - \beta_2 x_2)x \end{pmatrix}. \qquad \text{[Eq.4.1]}$$

If the matrix operator F satisfies Eq.4,

$$F^2x = -\omega^2x,$$

then we have Eqs.7-11,and therefore it follows that, by letting $a_{22} = -r'$ (= $\mathrm{r}_2 - \alpha_2 \mathrm{x}_1 - \beta_2 x_2$),

$$-r' = -a_{11} = -(\mathrm{r}_1 - \alpha_1 \mathrm{x}_1 - \beta_1 x_2), \qquad \text{[Eq.4.2]}$$

and

$$(\mathrm{r}_1 - \alpha_1 \mathrm{x}_1 - \beta_1 x_2)^2 + \omega^2 = aa' x_1 x_2, \qquad \text{[Eq.4.3a]}$$

the latter being rewritten as a quadratic equation:

$$g(x_1, x_2) = \alpha^2 x_1{}^2 + 2hx_1x_2 + \beta^2 x_2{}^2 - 2r_1(\alpha_1 \mathrm{x}_1 + \beta_1 x_2) + (r_1{}^2 - \omega^2) = 0, \qquad \text{[Eq.4.3b]}$$

where $\quad h = \alpha\beta - aa'$. $\qquad$ [Eq.4.3c]

Eq.4.3b represents a conic curve, which is a trajectory of Eq.4.1 satisfying $F^2x = -\omega^2x$.

Hessian matrix of $g(x_1,x_2)$ is

$$H = \begin{pmatrix} \alpha^2, & h, & -r_1\alpha \\ h, & \beta^2, & -r_1\beta \\ -r_1\alpha, & -r_1\beta, & r_1{}^2 - \omega^2 \end{pmatrix}. \qquad \text{[Eq.4.4]}$$

By letting

$$H_0 = \begin{pmatrix} \alpha^2, & h \\ h, & \beta^2 \end{pmatrix} = \begin{pmatrix} \alpha^2, & \alpha\beta - aa' \\ \alpha\beta - aa', & \beta^2 \end{pmatrix}.$$

and

$$L = (-r_1\alpha, -r_1\beta),$$

Eq.4.3b is written as

$$g(x_1,x_2) = {}^t x H_0 x + Lx + (r_1{}^2 - \omega^2) = 0, \qquad \text{[Eq.4.3c]}$$

and we then have

$$|H_0| = \alpha^2\beta^2 - h^2 = -aa'(2\alpha\beta - aa'). \qquad \text{[Eq.4.3d]}$$

By letting κ be eigenvalue of H_0, κ satisfies a simultaneous equation;

$(H_0-\kappa E)x = \begin{pmatrix} (\alpha^2-\kappa)x_1 + hx_2 \\ hx_1 + (\beta^2-\kappa)x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, [Eq.4.4]

and an eigenequation is

$$|H_0-\kappa E| = (\alpha^2-\kappa)(\beta^2-\kappa) - h^2 = 0,$$

i.e.,

$$\kappa^2 - (\alpha^2+\beta^2)\kappa + (\alpha^2\beta^2 - h^2) = 0.$$

[Eq.4.5]

Thus eigenvalues (κ) of H_0 are given below;

$\kappa_i = \{\alpha^2+\beta^2+\sigma_i D_o^{1/2}\}/2, \ (i = 1,2),$ [Eq.4.6]

where

$D_o = (\alpha^2 - \beta^2)^2 + 4h^2,$ [Eq.4.6a]

and $(\sigma_1, \sigma_2) = (1,-1).$ [Eq.4.6b]

Eigenvectors v_i ($= {}^t(v_{i1}, v_{i2})$) of H_0, satisfying

$v_{i1}^2 + v_{i2}^2 = 1,$ [Eq.4.7]

and corresponding to κ_i ($i=1,2$) are therefore found as

$v_i = \begin{pmatrix} \{(\kappa_i - \beta^2)/h^2\}t_i \\ t_i \end{pmatrix},$ [Eq.4.8]

where

$t_i = (\kappa_i D_0^{1/2}/h^2)^{-1/2},$

$(i=1,2), (\kappa_i > 0),$ [Eq.4.9]

which can be obtained as below;

$$\begin{aligned} t_i &= \{(\kappa_i - \beta^2)^2/h^2 + 1\}^{-1/2} \\ &= \{(\alpha^2-\beta^2+\sigma_i D_0^{1/2})^2/(4h^2) + 1\}^{-1/2} \\ &= [(\alpha^2-\beta^2+\sigma_i D_0^{1/2})^2+4h^2\}/(4h^2)]^{-1/2} \\ &= [\{(\alpha^2-\beta^2)^2+4h^2\}+D_0+2\sigma_i(\alpha^2-\beta^2)D_0^{1/2}\}/(4h^2)]^{-1/2} \\ &= [D_0+D_0+2\sigma_i(\alpha^2-\beta^2)D_0^{1/2}\}/(4h^2)]^{-1/2} \\ &= [\{(\alpha^2-\beta^2+\sigma_i D_0^{1/2})/2\}D_0^{1/2}/h^2]^{-1/2} \\ &= (\kappa_i D_0^{1/2}/h^2)^{-1/2}, \ (i=1,2). \end{aligned}$$

Thus we find an orthogonal matrix, P, for diagonalizing H_0, as below:

$P = (p_{ij}) = \begin{pmatrix} v_{11}, v_{21} \\ v_{12}, v_{22} \end{pmatrix}.$ [Eq.4.10]

Letting $X = {}^t(X_1,X_2)$, P satisfies

$x = PX.$ [Eq.4.11]

Thus we have

$$\begin{aligned} x_1 &= p_{11}X_1 + p_{12}X_2 \\ &= v_{11}X_1 + v_{21}X_2 \\ x_2 &= p_{21}X_1 + p_{22}X_2 \\ &= v_{12}X_1 + v_{22}X_2 \end{aligned}$$

Since $P^{-1} = {}^tP$, it follows that

$X = P^{-1}x = {}^tPx$ [Eq.4.12]

$= \begin{pmatrix} v_{11}, v_{12} \\ v_{21}, v_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= \begin{pmatrix} v_{11}x_1 + v_{12}x_2 \\ v_{21}x_1 + v_{22}x_2 \end{pmatrix}.$ [Eq.4.12a]

Accordingly, for $i = 1, 2$, we have

$X_i = v_{i1}x_1 + v_{i2}x_2$ [Eq.4.13]

$= \{(\kappa_i - \beta^2)/h\}t_i x_1 + t_i x_2$ [Eq.4.13a]

$= [\{(\alpha^2 - \beta^2 + \sigma_1\sqrt{D_0})/(2h)\}x_1 + x_2]t_i$

$= \pm[\{(\alpha^2 - \beta^2 + \sigma_1\sqrt{D_0})/(2h)\}x_1 + x_2]$

$/\sqrt{\kappa_i\sqrt{D_0}/h^2}).$

[Eq.4.13b]

Furthermore, we find

$LPX = (-r_1\alpha, -r_1\beta)\begin{pmatrix} v_{11}, v_{21} \\ v_{12}, v_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$= (-r_1\alpha, -r_1\beta)\begin{pmatrix} v_{11}x_1 + v_{21}x_2 \\ v_{12}x_1 + v_{22}x_2 \end{pmatrix}$

$= -r_{1\{(}\alpha v_{11} + \beta v_{12})x_1 + {}_{(}\alpha v_{21} + \beta v_{22})x_{2\}}$

[Eq.4.14]

By letting $K = {}^t((\kappa_1,0),(0, \kappa_2))$, now we have

${}^tXKX+LPX+(r_1^2 - \omega^2) = 0,$ [Eq.4.15]

meaning that

$\kappa_1 X_1^2 + \kappa_2 X_2^2$

$- r_1\{(\alpha v_{11} + \beta v_{21})x_1 + (\alpha v_{12} + \beta v_{22})x_2\} + (r_1^2$

$+\omega^2) = 0.$ [Eq.4.15a]

Based on the above analyses, this equation is rewritten by

$(X_1 - X_{10})^2/(C/\kappa_1) + (X_2 - X_{20})^2/(C/\kappa_2) = 1$

[Eq.4.16]

where

$X_{10} = r_1(\alpha v_{11} + \beta v_{21})/(2\kappa_1),$

$X_{20} = r_1(\alpha v_{12} + \beta v_{22})/(2\kappa_2),$

$C = (r_1^2/4)\{(\alpha v_{11} + \beta v_{21})^2/\kappa_1$

$+ (\alpha v_{12} + \beta v_{22})^2/\kappa_2\} - (r_1^2 + \omega^2).$

Since $\kappa_1\kappa_2 > 0$, if ω does not depends neither on t nor on x_i, this equation (Eq.4.16) represents an ellipse with semi-axis' lengths, $(C/\kappa_1)^{1/2}$ and $(C/\kappa_2)^{1/2}$, when $C > 0$, and a hyperbola when $C < 0$. These conic curves are trajectories represented by (X_1,X_2) which are obtained by the orthogonal transformation, Eq.4.12a, from the trajectories represented by Eq.4.3b. Thus both Eq.4.3b and Eq.4.16 represents the same conic curve being a trajectory of Eq.4.1 satisfying $F^2x = -\omega^2x$. If $C > 0$, this conic curve trajectory is an ellipse, meaning that $F^2x = -\omega^2x$ represents a harmonic oscillation. If $C < 0$, on the other hand, the corresponding trajectory is a hyperbola, meaning that harmonic oscillation does not occur..

In more general cases where C (> 0) depends on t (*i.e.*, $C= C(t)$), the long and short diameters of the ellipse varies depending on t, *confirming that* X_1 *and* X_2 (and therefore, x_1 and x_2) give *a generalized harmonic oscillation.*

5. Harmonic oscillations in Schrödinger equation

This section describes something about the harmonic oscillation in Schrödinger equation, from a viewpoint of the above-mentioned matrix operator equation.

Schrödinger equation is given by

$$-i\hbar\partial\psi/\partial t = -\{\hbar^2/(2m)\}\partial^2\psi/\partial x^2 + V(x)\psi \quad \text{[Eq.6.1]}$$

As is well-known, by letting

$$\psi(x,t) = u(x)f(t),$$

$\psi(x,t)$ can be separated to a t-dependent portion $f(t)$ and an x-dependent portion $u(x)$, satisfying

$$\frac{i\hbar}{f(t)}\frac{df(t)}{dt} = \frac{1}{u(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2u(x)}{dx^2} + V(x)u(x)\right) = E \quad \text{[Eq.6.2]}$$

where E is a constant. It therefore follows that

$$i\hbar\frac{df(t)}{dt} = Ef(t)\,. \quad \text{[Eq.6.3]}$$

Eq.6.3 is rewritten as

$$Df(t) = -i(E/\hbar)f(t) = -i\omega f(t) \quad \text{[Eq.6.4a]}$$

in which $D=d/dt$ and $\omega = E/\hbar$, and further written as;

$$(D + i\omega)f(t) = 0\,. \quad \text{[Eq.6.4b]}$$

By using another operator, $(D + i\omega)$, we have

$$(D - i\omega)(D + i\omega)f(t) = 0, \quad \text{[Eq.6.5a]}$$

meaning that

$$(D^2 + \omega^2)f(t) = 0, \quad \text{[Eq.6.5b]}$$

or,

$$d^2f(t)/dt^2 = -\omega^2 f(t)\,. \quad \text{[Eq.6.5c]}$$

From Theorem 1 and Eq.21b (in Section 2), if we use $F = (a_{ij})$ hitherto discussed and $f_0(t) = {}^t(f_1(t),\ f_2(t))$, Eq.6.5c means that, under the condition of $(da_{12}/dt)(da_{21}/dt)=-(da_{11}/dt)^2$ (Eq.21b), (e.g., a_{ij} are constants), we have

$$D\,f_0(t) = F\,f_0(t), \quad \text{[Eq.6.7a]}$$

$$F^2f_0(t) = \begin{pmatrix} a_{11}, & a_{12} \\ -\dfrac{a_{11}{}^2+\omega^2}{a_{12}}, & -a_{11} \end{pmatrix}^2 \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = -\omega^2\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}. \quad \text{[Eq.6.7b]}$$

Thus Eqs.6.7a/b elucidates that

$$d^2f_1(t)/dt^2 = -\omega^2 f_1(t), \quad \text{[Eq.6.8a]}$$

$$d^2f_2(t)/dt^2 = -\omega^2 f_2(t)\,. \quad \text{[Eq.6.8b]}$$

Experimental data satisfying Eq.6.4a have hitherto been accumulated during the long history of quantum mechanics, which suggests Eqs. 6.5a/b, whose solutions are $f_1(t)$ and $f_2(t)$ in Eqs.6.8a/b. $f_1(t)$ is considered to be related to the probability of the existence of quantum element, whereas $f_2(t)$ does not seem to have been directly analyzed. Theoretical analyses mentioned above in this paper seem to suggest that we may need to find what $f_2(t)$ could really be. Could $f_2(t)$ be related to some unknown matter or element or some unknown phenomenon other than those we presently know ? The relationship between F and the so-called "spin matrix" or something like might have some essence for answering this question.

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