Relations between Network Reliability and Number of Trees of Graph

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Abstract

Network reliability is the probability that all stations of the network are connected by only unfailing links under independent link failures. The communication network can be represented by an undirected graph G, where communication stations and links are corresponding to nodes and branches of its graph, respectively. Therefore, network reliability can be defined as the probability that there exists at least one tree in the corresponding graph. We assume that each branch connects its two end nodes with the same independent probability p and disconnects with probability q=1- p. In case of p 1 or p 0, it is known that the most reliable network has the maximum number of trees within graphs of n nodes and b branches.

In this paper, we present a network reliability Pr(G) and a number of trees t(G) of graph G by using the generation algorithm of exclusive events which include trees. Then given two graphs G_A and G_B with same numbers of nodes n, branches b, and different number of trees $t(G_A)<t(G_B)$, we demonstrate network reliabilities $Pr(G_A)<Pr(G_B)$ with respect to probabilities $p(0<p<1)$ in case of smaller graphs with n=4, b=4 and n=5, b=7.

Keywords: Network reliability, Number of trees, Exclusive events

1 Introduction

If we focus on relations of connection between stations by links, the communication network can be represented by an undirected graph G, where communication stations and links are corresponding to nodes and branches of its graph, respectively. A graph denoted by $G=(V,E)$ is composed with a node set V and an edge set E. For simplicity, graph G is assumed a simple graph which is an undirected graph with no self-loops and no multiple branches.

Network reliability is defined as the probability that all stations of the network are connected by only unfailing links under independent link failures. Then the network reliability Pr(G) of a graph G is a probability that there exists at least one tree composed with only normal branches.[1],[2] Namely, if we can obtain all trees of graph G as T1, T_2,・・・,T_k, then Pr(G) would be represented by expression (1) below.

$$Pr(\mathrm{G})= Pr(\mathrm{T}_1\cup \mathrm{T}_2\cup\cdots,\cup \mathrm{T}_k) \qquad (1)$$

It is not an efficient method to expand expression (1) to calculate Pr(G) for a large graph G with relatively many trees k= t(G). To avoid these tedious operations, the procedure was proposed that generates exclusive events including trees and cuts of graph at the same time.[1] This procedure can also calculate probability Pr(G) within an arbitrary error. Now, assuming each branch connects its two end nodes with independent, equal probability p and disconnects with probability q=1-p, it is known that the most reliable network has the maximum number of trees in case of p 1 or p 0.[2]

We first present a procedure to estimate the number of trees t(G) and the network reliability Pr(G) of graph G by using the generation algorithm of exclusive events including trees of graph G.

Next, given two graphs G_A and G_B with same numbers of nodes n, branches b, and different number of trees $t(G_A)<t(G_B)$, we demonstrate network reliabilities $Pr(G_A)<Pr(G_B)$ with respect to probabilities $p(0<p<1)$ in case of smaller graphs with n=4, b=4 and n=5, b=7.

2 Preliminary

First, we introduce some graph theoretical notations. A graph denoted by G is composed with a node set and a branch set. In graph G, we denote an event of existing branch e by e, and an event of not existing branch e by $\bar{e}$. Union and product of events are denoted by + and ・, respectively. Then two events τ_i and τ_j ($i \neq j$) that include at least one tree of G are said to be exclusive events if $\tau_i \cdot \tau_j = \phi$ (empty event). Union event T_X [G] of exclusive tree events including all trees of G can be generated by next expression (2) with respect to a multiple branch { $e_1, e_2, \cdots, e_k$ } of $k(k \geqq 1)$ branches.[1]

$$T_X[\mathrm{G}]=(e_1+e_2+\cdots+e_k)\cdot T_X[\mathrm{G}(e_1+e_2+\cdots+e_k)] \oplus (\bar{e}_1\cdot\bar{e}_2\cdots\bar{e}_k)\cdot T_X[\mathrm{G}(\bar{e}_1\cdot\bar{e}_2\cdot\ \cdots\ \cdot\bar{e}_k)] \qquad (2)$$

where, G($e_1+e_2+\cdots+e_k$) and G($\bar{e}_1\cdot\bar{e}_2\cdot\ \cdots\ \cdot\bar{e}_k$) are graphs derived from G by short circuiting and open circuiting all branches of multiple branch {$e_1, e_2, \ldots, e_k$}, respectively. $\oplus$ denotes direct sum of exclusive events. As the result of repeating (2), we have obtained next expression (3).

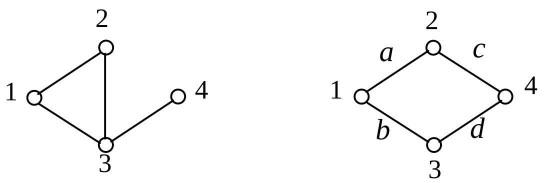
$$\left.\begin{aligned} &T_X[\mathrm{G}] = \tau_1 \oplus \tau_2 \oplus \cdots \oplus \tau_{t_X} \\ &\tau_i = \prod_{j=1}^{n-1}(e_{ij1}+e_{ij2}+\cdots+e_{ijk}) \\ &\quad\cdot \prod_{\ell=1}^{s}(\bar{e}_{i\ell 1}\cdot\bar{e}_{i\ell 2}\cdot\cdots\cdot\bar{e}_{i\ell m})\ ;\ i=1,2,\cdots,t_X \end{aligned}\right\} \quad (3)$$

where, n denotes the number of nodes, s is the number of multiple branches short circuited in j-th order and open circuited in ℓ-th order respectively, and t_X is the number of exclusive tree event. Since simultaneous occurrence probability $Pr(\tau_i\cdot\tau_j\cdots\tau_k)=0$ ($i\neq j\neq\cdots\neq k$) with respect to these τ_i's, network reliability Pr(G) result in (4).

$$\left.\begin{aligned} &R(\mathrm{G}) = Pr(T_X[\mathrm{G}]) = Pr\Big(\bigcup_{i=1}^{t_X}\tau_i\Big) = \sum_{i=1}^{t_X} Pr(\tau_i) \\ &Pr(\tau_i) = \prod_{j=1}^{n-1}\Big(1-\prod_{h=1}^{k}(1-p_{ijh})\Big)\cdot\prod_{\ell=1}^{s}\Big(\prod_{h=1}^{m}(1-p_{i\ell h})\Big) \end{aligned}\right\} (4)$$

3 Number of trees and network reliability

Figure 1 shows two graphs G_1 and G_2 with same numbers of nodes n=4 and branches b=4.



(a) G_1 (b) G_2

Fig.1. Graphs n=4, b=5.

Figure 2 represents the process of generating exclusive tree events τ_i using (2) with respect to (multiple) branch incident to a node in the order of 1,2,$\cdots$ of graph G_1 in Fig.1(a), where each branch is discriminated by symbols a, b, c and d. As shown in Fig.2, graph G_1 has t_X=2 exclusive tree events τ_1 and τ_2, where τ_1 and τ_2 include two and one trees, respectively. Then the total number of trees $t(G_1)$=3. Now if all of four branches of G_1 operate normally by the same probability p(q=1-p), the network reliability $R(G_1)$ of corresponding graph G_1 can be calculated into

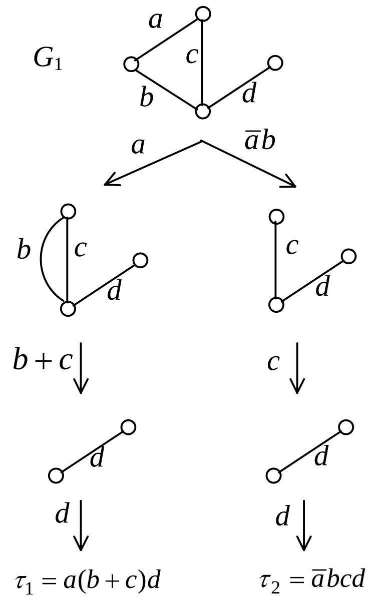


Fig.2 Generation of exclusive tree events.

expression (5) using (3) and (4).

$$R(G_1) = Pr(T_X[G_1]) = Pr\Big(\bigcup_{i=1}^{t_X}\tau_i\Big)$$
$$= p^2(1-q^2)+p^3q \quad (5)$$

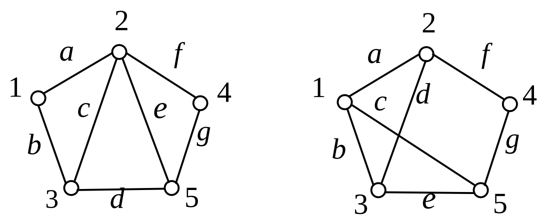
For graph G_2 in Fig.1(b), next three exclusive tree events were derived in similar way.

$$\tau_1 = ab(c+d)$$
$$\tau_2 = a\bar{b}dc$$
$$\tau_3 = \bar{a}bdc$$

As a result of these exclusive events, the number of trees is $t(G_2)$=4. For equal branch probability p (q=1-p), network reliability was obtained as (6).

$$R(G_2) = p^2(1-q^2)+2p^3q \quad (6)$$

Figure 3 shows two graphs G_3 and G_4 with n=5, b=7.



(a) G_3 (b) G_4

Fig.3 . Graphs with n=5, b=7.

In the similar way as graph G_1, there are 8 exclusive tree events that each includes 1~8 trees of G_3. There are 10 exclusive tree events that each includes 1~6 trees of G_4 Then the number of trees is $t(G_3)=21$ and $t(G_4)=24$. For equal branch probability p (q=1-p), network reliability was obtained as (7) and (8).

$$R(G_3)=2p^4q^3+4p^3(1-q^2)q^2+p(1-q^2)^3 \\ +p^3(1-q^3)q \qquad (7)$$

$$R(G_4)=3p^4q^3+4p^3(1-q^2)q^2+p^2(1-q^2)(1-q^3) \\ +p^2(1-q^2)^2q+p^3(1-q^3)q \qquad (8)$$

Comparing (5),(6) with (7),(8), it is observed that increase of nodes and branches makes reliability expressions rapidly complicated.

In case of p 1 or p 0, the necessary condition that the network is most reliable becomes the graph to have the maximum number of trees. If the network with the maximum number of trees is unique, the above necessary condition becomes also sufficient condition.[2]

We compare the network reliability R(G) with respect to branch probability p=0.1～0.9 for graphs of same numbers of nodes n and branches b used in Figs. 1 and 3.

The result is shown in table 1.

Table 1. Network reliability and the number of trees.

nodes n, branches b	n=4, b=4		n=5, b=7	
graph	G_1	G_2	G_3	G_4
trees t(G)	3	4	21	24
Network relia. R(G) / branch prob. p	$R(G_1)$	$R(G_2)$	$R(G_3)$	$R(G_4)$
0.9	0.8748	0.9639	0.90345	0.98415
0.8	0.7168	0.8704	0.79299	0.91750
0.7	0.5488	0.7399	0.65283	0.78753
0.6	0.3888	0.5904	0.48833	0.60653
0.5	0.25	0.4375	0.32031	0.40625
0.4	0.1408	0.2944	0.17490	0.22528
0.3	0.0648	0.1719	0.07234	0.09428
0.2	0.0208	0.0784	0.01834	0.02413
0.1	0.0028	0.0199	0.00145	0.00192

Table 1 shows that graphs G_2 and G_4 which have larger number of trees give higher reliability compared with graphs G_1 and G_3.

4. Conclusion

Analytical expressions for network reliability are presented using exclusive tree events for smaller number of nodes and branches. The number of trees would play an important role for network reliability. Since the number of trees becomes huge for large graph, it is tedious to obtain analytical expressions of reliability. Then numerical computation, for example the number of trees become useful.[3]

References

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Appendix

Exclusive tree events for graph G_3

$$\tau_1=a(b+c)(d+e)(f+g)$$
$$\tau_2=a(b+c)\bar{d}\bar{e}fg$$
$$\tau_3=a\bar{b}\bar{c}ed(f+g)$$
$$\tau_4=a\bar{b}\bar{c}\bar{e}fdg$$
$$\tau_5=\bar{a}bcf(d+e+g)$$
$$\tau_6=\bar{a}bc\bar{f}(e+d)g$$
$$\tau_7=\bar{a}b\bar{c}fd(e+g)$$
$$\tau_8=\bar{a}b\bar{c}\bar{f}edg$$

Exclusive tree events for graph G_4

$$\tau_1=a(b+d)f(c+e+g)$$
$$\tau_2=a(b+d)\bar{f}(e+c)g$$
$$\tau_3=a\bar{b}\bar{d}f(c+g)e$$
$$\tau_4=a\bar{b}\overline{df}ceg$$
$$\tau_5=\bar{a}bdf(c+e+g)$$
$$\tau_6=\bar{a}bd\bar{f}(c+e)g$$
$$\tau_7=\bar{a}b\bar{d}f(c+e)g$$
$$\tau_8=\bar{a}\bar{b}cde(f+g)$$
$$\tau_9=\bar{a}\bar{b}cd\bar{e}fg$$
$$\tau_{10}=\bar{a}\bar{b}c\bar{d}fge$$