Phase Synchronization of Limit Cycle Oscillators in a Fluctuating Environment

Koh Hashimoto[*1,*] and Kazuyuki Aihara[*1,*2,*3]
[*1] Aihara Complex Modelling Project, ERATO, JST
Room M203, Komaba Open Laboratory, The University of Tokyo,
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, JAPAN
[*2] Institute of Industrial Science, The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, JAPAN
[*3] Graduate School of Information Science and Technology, University of Tokyo
7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, JAPAN
[*] koh@aihara.jst.go.jp

Abstract

We study the synchronization of limit cycle oscillators in a fluctuating environment. When environmental conditions fluctuate due to various kinds of noise, the dynamics of elements in the environment are inevitably perturbed and this may cause some synchronization between them. We analyze this phenomenon using models in which system parameters are subject to external noise and fluctuate within a certain range. Using the phase reduction method, we discuss the synchrony of limit cycle oscillators and show that the Lyapunov exponent is negative when amplitude of noise is sufficiently small, namely, synchronization of limit cycle oscillators in a fluctuating environment is stable.

1 introduction

Phase synchronization of limit cycle oscillators is a ubiquitous phenomenon, found in a variety of biological, chemical and physical fields, and has attracted much attention for several decades since Winfree's pioneering work [1] in 1960s. Populations of limit cycle oscillators that are subject to a strong periodic force may be entrained to oscillate at the same frequency as the periodic driving force. Alternatively, limit cycle oscillators interacting with each other directly or indirectly can synchronize precisely due to mutual interactions among oscillators. In either case, external or internal noise sources may disturb the phase synchronization of the oscillators because the main effect of noise on oscillation is phase diffusion. And therefore noise has long been considered to exert a negative influence on synchronization of limit cycle oscillators.

However noise can play also an active role in synchronization of non-interacting oscillators. Challenging works by Teramae and Tanaka [2] and Goldobin and Pikovsky [3] have shown that oscillators under the influence of common weak additive noise can synchronize regardless of their intrinsic properties and the initial conditions. Since this noise-driven synchronization does not depend on the natural frequency of oscillators, it obviously differs from entrainment to an external periodic force. Using the phase reduction method, they proved in general that the maximal Lyapunov exponent of an orbit is always negative with little constraints when they are subject to weak Gaussian-white noise and this means the phase synchronization of the oscillators.

In their works, noise perturbs states of elements in the system directly and amplitude of fluctuations are identical for all elements. However, in some natural systems, the influence of noise on elements should be construed in a different way. In a variety of natural systems, environmental conditions such as temperature and pressure, which determine the dynamics of elements in the systems, are perturbed by various kinds of noise sources. In such systems, noise does not perturb system states directly but dynamics of systems. Under such an environment, dynamical elements, which exhibit oscillation, are perturbed indirectly by those kinds of noise sources and amplitude of fluctuations are depend on the states of elements. Thus we investigate behaviors of non-coupling limit cycle oscillators in a fluctuating environment and show another scenario to reach synchronization. In a mathematical model, we can treat this situation by adding noise on system parameters instead of system states. Using the phase reduction method which is applicable to an arbitrary oscillator [4, 5], we analytically calculate the Lyapunov exponent of the synchronizing state and prove that the exponent is negative with some reasonable restrictions. With system parameters fluctuating within a structurally stable region under influence of external weak noise, phase synchronization of limit cycle oscillators can occur.

2 Model Description

Population of N identical nonlinear oscillators with fluctuating parameters is described as

$$\dot{\boldsymbol{x}}^{(i)} = \boldsymbol{F}(\boldsymbol{\mu}; \boldsymbol{x}^{(i)}) \tag{1}$$

$$\dot{\boldsymbol{\mu}} = \boldsymbol{\eta}(\boldsymbol{\mu}, t) = -\nabla U(\boldsymbol{\mu}) + \boldsymbol{\xi}(t) \tag{2}$$

where $i = 1, \ldots, N$ and $\boldsymbol{x}^{(i)}$ is a state vector of the i-th element in this system. $\boldsymbol{F}$ is common dynamics of the elements and $\boldsymbol{\mu}$ is a parameter vector of the function $\boldsymbol{F}$. $\boldsymbol{\mu}$ has its energy function U and is perturbed by noise $\boldsymbol{\xi}$. $\boldsymbol{\xi}$ is a vector of Gaussian white noise. The elements of the vector are normalized as $\langle \xi_k(t) \rangle = 0$ and $\langle \xi_k(s)\xi_l(t) \rangle = 2D_{kl}\delta(s-t)$, where $\boldsymbol{D} = (D_{kl})$ is a variance matrix of the noise components.

We assume that:
(a) $\boldsymbol{\mu}$ is bounded within a bounded domain by the energy function U with probability 1,
(b) F has no bifurcation in the domain,
(c) F has a limit cycle attractor $C(\boldsymbol{\mu})$ in the domain,
(d) F is continuously differentiable by $\boldsymbol{\mu}$ and $\boldsymbol{x}$,
(e) D_{kl} is sufficiently small.
For these assumptions, we can assume that an attractor of F is always a limit cycle, which varies continuously with changing of the parameter vector $\boldsymbol{\mu}$, and that a state of an element is always sufficiently close to the limit cycle $C(\boldsymbol{\mu})$.

3 Reduction To Phase Dynamics

Just as in the previous works, we use the phase reduction method to analyze this system. However, in this system, the limit cycle, on which the elements are, varies constantly according to fluctuations of the parameters. And this makes it difficult to define a phase for an element. Thus, we should make some preparations for phase reduction.

At first, with constant parameter vector $\boldsymbol{\mu}$, we can define a phase for a point on $C(\boldsymbol{\mu})$ following standard procedure [4, 5]. In this article, we normalize phase by the period of the limit cycle $C(\boldsymbol{\mu})$ so that its range is [0,1], where 0 and 1 represents the same phase. We represent a phase θ for a point $\boldsymbol{x} \in C(\boldsymbol{\mu})$ with constant parameter vector $\boldsymbol{\mu}$ by $\theta = \Theta_{\boldsymbol{\mu}}(\boldsymbol{x})$ and its reverse function by $\boldsymbol{x} = \boldsymbol{X}_{\boldsymbol{\mu}}(\theta)$. If the parameter vector $\boldsymbol{\mu}$ is constant, phase dynamics are simply written as $\dot{\theta}^{(i)} = \omega(\boldsymbol{\mu})$. Note that the zero phase point $X_{\boldsymbol{\mu}}(0)(= X_{\boldsymbol{\mu}}(1))$ can be chosen arbitrarily.

Secondly, a phase for a point in neighborhood of $C(\boldsymbol{\mu})$ can be defined using an isochrone of a point on $C(\boldsymbol{\mu})$, i.e., identify a point $\boldsymbol{x}' \notin C(\boldsymbol{\mu})$ to a point $\boldsymbol{x} \in C(\boldsymbol{\mu})$ in a way that the orbits from the two points asymptotically coincide with the parameter vector fixed to $\boldsymbol{\mu}$ (see Fig.1), we represent this map from a point $\boldsymbol{x}' \notin C(\boldsymbol{\mu})$ to a point $\boldsymbol{x} \in C(\boldsymbol{\mu})$ by $\boldsymbol{x} = \Psi_{\boldsymbol{\mu}}(\boldsymbol{x}')$, and let the phase of $\boldsymbol{x}'$ be the phase of $\boldsymbol{x} = \Psi_{\boldsymbol{\mu}}(\boldsymbol{x}')$ with constant parameter $\boldsymbol{\mu}$.

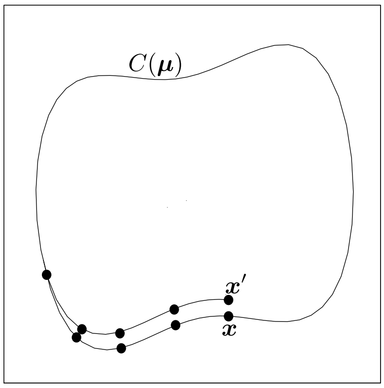

Figure 1: When the parameter vector is fixed to $\boldsymbol{\mu}$, a phase of a point $\boldsymbol{x}'$ in a neighborhood of $C(\boldsymbol{\mu})$ can be defined by identifying its phase to a phase of a point $\boldsymbol{x}$ which satisfies that the orbit from $\boldsymbol{x}$ asymptotically coincide with the one from $\boldsymbol{x}'$.

When the parameter vector varies from $\boldsymbol{\mu}$ to $\boldsymbol{\mu}+\Delta\boldsymbol{\mu}$ at time t, a phase of an element varies according to changing of the attractor $C(\boldsymbol{\mu}) \to C(\boldsymbol{\mu} + \Delta\boldsymbol{\mu})$. A map from a phase with $\boldsymbol{\mu}$ to a phase with $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ that describes phase slipping caused by varying the parameters at time t can be defined as

$$\theta' = \Phi_{\boldsymbol{\mu},\Delta\boldsymbol{\mu}}(\theta) = \Theta_{\boldsymbol{\mu}+\Delta\boldsymbol{\mu}}(\Psi_{\boldsymbol{\mu}+\Delta\boldsymbol{\mu}}(X_{\boldsymbol{\mu}}(\theta))). \tag{3}$$

Fig.2 is an example of Φ. Note that, because the zero phase point can be chosen arbitrarily as mentioned above, we can always align the phase for $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ to satisfy $\Phi_{\boldsymbol{\mu},\Delta\boldsymbol{\mu}}(0) = 0$ $(\Phi_{\boldsymbol{\mu},\Delta\boldsymbol{\mu}}(1) = 1)$ as we see in Fig.2. With this alignment, a value $\Delta s = \Phi_{\boldsymbol{\mu},\Delta\boldsymbol{\mu}}(\theta) - \theta$ means a phase shift caused by changing of the parameter vector $\boldsymbol{\mu} \to \boldsymbol{\mu} + \Delta\boldsymbol{\mu}$.

Imagine that the variation of the parameter vector $\boldsymbol{\mu} \to \boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ is occurred continuously during Δt

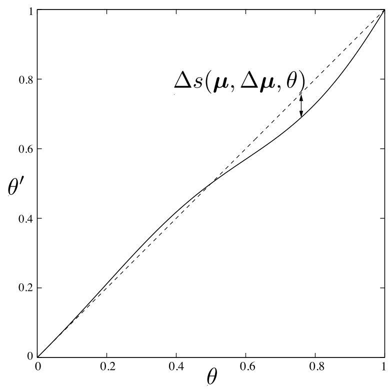

Figure 2: If the parameter vector varies from $\boldsymbol{\mu}$ to $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ at time t, a phase of each point is redefined by Eq.(3). Origin of phase for $\boldsymbol{\mu} + \Delta\boldsymbol{\mu}$ is aligned so that $\Phi_{\boldsymbol{\mu},\Delta\boldsymbol{\mu}}(0) = 0$ ($\Phi_{\boldsymbol{\mu},\Delta\boldsymbol{\mu}}(1) = 1$) is satisfied. $\Delta s = \theta' - \theta$ means a phase shift caused by changing of the parameters.

without moving an element $\boldsymbol{x}$ by F in order to extract only the effect of phase shift from dynamics. Now we should define a phase shift function not for $\boldsymbol{\mu}$ and $\Delta\boldsymbol{\mu}$ but for $\boldsymbol{\mu}$ and $\dot{\boldsymbol{\mu}}$. And this is derived by taking a limit of Δt to 0 as

$$\phi(\boldsymbol{\mu}, \dot{\boldsymbol{\mu}}, \theta) = \lim_{\Delta t \to 0} \frac{\Delta s(\boldsymbol{\mu}, \Delta\boldsymbol{\mu}, \theta)}{\Delta t}. \tag{4}$$

Using Eq.(4), we can reduce Eq.(1) as following:

$$\dot{\theta} = \omega(\boldsymbol{\mu}) + \phi(\boldsymbol{\mu}, \dot{\boldsymbol{\mu}}, \theta) = \omega(\boldsymbol{\mu}) + \phi(\boldsymbol{\mu}, \boldsymbol{\eta}, \theta) \tag{5}$$

where $\omega(\boldsymbol{\mu})$ is a rotating velocity term determined by $\boldsymbol{\mu}$ and F, and $\phi(\boldsymbol{\mu}, \boldsymbol{\eta}, \theta)$ is a phase shift term determined by $\boldsymbol{\mu}, \dot{\boldsymbol{\mu}}$ and F. In fact, this reduction is valid only when $\omega(\boldsymbol{\mu})$ is sufficiently larger than $\phi(\boldsymbol{\mu}, \boldsymbol{\eta}, \theta)$ and the assumption (e) that we have in the previous section ensures this.

4 Phase Synchronization Induced By Fluctuating Environment

Suppose that the two phases have an infinitesimally small difference $\Delta\theta = \theta_2 - \theta_1$ where θ_i obeys Eq.(5). Then the Lyapunov exponent is defined as the long time average of $\dfrac{\mathrm{d}}{\mathrm{d}t} \log \Delta\theta$. By replacing the long time average with the ensemble average with respect to $\boldsymbol{\xi}$, we can represent the Lyapunov exponent as

$$\lambda = \left\langle \frac{\mathrm{d}}{\mathrm{d}t} \log \Delta\theta \right\rangle_{\boldsymbol{\xi}}.$$

With following additional assumptions:
(f) ϕ is second-times continuously differentiable by θ,
(g) ϕ is continuously differentiable by $\boldsymbol{\mu}$ and $\boldsymbol{\eta}$,
we can obtain the following formula:

$$\lambda = -\int_0^1 \mathrm{d}\theta \int \mathrm{d}P(\boldsymbol{\mu}) \sum_{k,l} D_{kl} \frac{\partial \phi'(\boldsymbol{\mu}, \mathbf{0}, \theta)}{\partial \eta_k} \frac{\partial \phi'(\boldsymbol{\mu}, \mathbf{0}, \theta)}{\partial \eta_l} \tag{6}$$

where ϕ' means $\phi' = \dfrac{\partial \phi}{\partial \theta}$ and $P(\boldsymbol{\mu})$ is a steady distribution function of $\boldsymbol{\mu}$.

We have the last assumption here:
(i) $\phi(\boldsymbol{\mu}, \boldsymbol{\eta}, \theta) \neq 0$ for almost every $(\boldsymbol{\mu}, \boldsymbol{\eta}, \theta)$.
This assumption means that fluctuation of parameters almost always causes phase shift. Since this assumption ensures $\frac{\partial \phi'(\boldsymbol{\mu}, \mathbf{0}, \theta)}{\partial \eta_k} \neq 0$ for almost every $(\boldsymbol{\mu}, \boldsymbol{\eta}, \theta)$ and the variance matrix D_{kl} is always positive definite, λ is negative. This means that the phase synchronization induced by perturbation of system parameters is stable in an arbitrary oscillator system with the assumptions we have.

5 Simulation

In this section, we demonstrate that phase synchronization of limit cycle oscillators can occur when noise strength is sufficiently small by numerical simulation using van der Pol oscillator and measure the Lyapunov exponents numerically.

Dynamics of van der Pol oscillator is described as

$$\ddot{x} = \gamma(1 - x^2)\dot{x} - x - bx^3$$

where γ and b are system parameters. Within a certain region of (γ, b), this system has a structurally stable limit cycle attractor. This differential equation can be rewritten in following form.

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \gamma(1 - x_1^2)x_2 - x_1 - bx_1^3 \end{cases} \tag{7}$$

In order to implement a fluctuating environment, we regard all coefficients in the terms in these differential equations as parameters (μ_k) and attach some additional terms $(\mu_1, \mu_3 x_2^2, \mu_8 x_1^2)$ as follows:

$$\begin{cases} \dot{x}_1 = \mu_1 + \mu_2 x_2 + \mu_3 x_2^2 \\ \dot{x}_2 = \mu_4(\mu_5 - \mu_6 x_1^2)x_2 - \mu_7 x_1 - \mu_8 x_1^2 - \mu_9 x_1^3 \end{cases}$$

And we adopt a "U-shape" function $U(\boldsymbol{\mu}) = \sum_k U_k(\boldsymbol{\mu})$ with

$$U_k(\boldsymbol{\mu}) = \begin{cases} 0 & (|\mu_k(t) - \mu_k(0)| < 0.05) \\ 2.5(\mu_k(t) - \mu_k(0))^4 & (\text{otherwise}) \end{cases}$$

for the energy function of $\boldsymbol{\mu}$. Initially, the parameters are set as: $\mu_1 = \mu_3 = \mu_8 = 0, \mu_2 = \mu_5 = \mu_6 = \mu_7 = 1, \mu_4 = \gamma, \mu_9 = b$ so that dynamics at the initial time is equivalent to the original equations Eq.(7). The simulation results are shown in Fig.3. At the initial time, the elements in the system are not synchronized at all. However, after long transient, they reach synchronization almost completely.

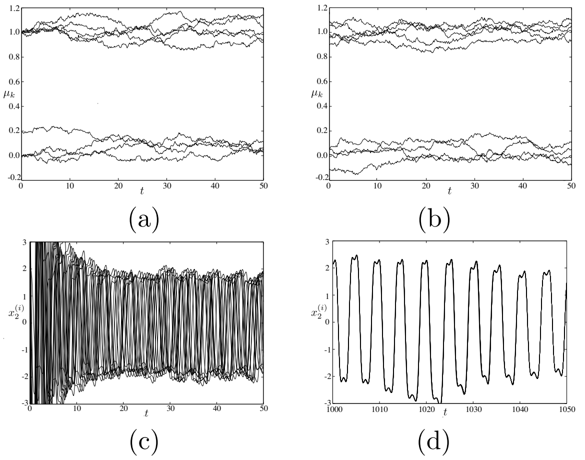

Figure 3: (a) and (b) show temporal evolution of parameters $\mu_1, \ldots, \mu_9$ for $[0, 50]$ and $[1000, 1050]$ respectively. They fluctuate all along the time by noise although bounded by U. Temporal evolution of x_2 of 16 orbits which start from points randomly chosen is plotted in (c) and (d). The variance matrix of the noise is set as $D_{kk} = 0.01, D_{kl} = 0 (k \neq l)$. The parameters γ and b are $\gamma = 0.2$ and $b = 1$.

Fig.4 shows the Lyapunov exponents that are numerically calculated for various noise strengths. When noise strength is smaller than a certain value, λ decreases linearly with the increase of the noise strength as indicated by Eq.(6). Meanwhile, when noise strength is strong, λ increases with the increase of the noise strength and too strong noise eventually destabilize the synchronization of oscillators and the Lyapunov exponent is no longer negative.

6 Summary

We analyzed phase synchronization induced by perturbation of system parameters by reducing the dynamics to phase dynamics. And we proved that when noise that perturb parameters are sufficiently weak and perturbation of parameters almost always causes phase shift, the Lyapunov exponent becomes negative.

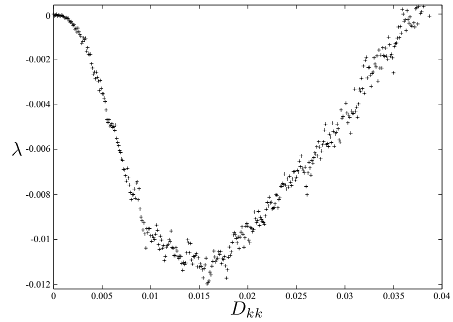

Figure 4: The horizontal axis is D_{kk} ($D_{kl} = 0$) and the vertical axis is the Lyapunov exponent λ. Each point is obtained by taking an average of 25 trials.

This result is achieved regardless of details of dynamics and initial distributions of elements.

In this article, we only treated the case in which parameters fluctuate continuously under influence of noise. Nagai and Nakao [6] discussed phase synchronization induced by a fluctuating input which jumps between two values at random moments and proved that when intervals of the jumps are sufficiently large and phase shift map is monotonic, the Lyapunov exponent of the system becomes negative. Using their ideas, our model is also applicable to the case in which parameters are perturbed discontinuously by noise. Studies for this case will be reported in the future.

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