An LMI approach to observer-based guaranteed cost control

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Abstract

In this paper, we present the design of observer-based guaranteed cost controller for a class of uncertain linear systems, in which full state cannot be measured. The perturbations are assumed to be described by structural uncertainties. Linear matrix inequality (LMI) approach is used to design the observer-based controller. The controller and observer gains are given from LMI optimization and feasibility problems, respectively. A numerical example shows the potential of the proposed method.

1 Introduction

During the last decades, considerable attention has been directed to the problem of robust stability analysis and robust stabilization of systems with parameter uncertainties. Recently, in addition to the simple stabilization, there has been much effort to design a controller which not only achieves the stability of the uncertain system but also guarantees an adequate level of performance. One approach to this problem is the guaranteed cost control method originally introduced by Chang and Peng [1]. Although the controller is usually constructed by using state variables, it may not be possible to measure all the states of the system in many cases [2,3]. Therefore, the problem of designing an observer-based guaranteed cost controller has received some attention in recent years. However, the algorithm presented by Lien cannot be implemented by the LMI control toolbox of MATLAB because it contains the equality condition [4]. Mahmoud *et al.* deal with the case where both the controller gain and the observer gain have prespecified forms, and they don't discuss the reduction of the performance index [5]. This paper deals with the design method which doesn't restrict the type of the observer gain and further achieves the reduction of the performance index. This method can be implemented by the LMI control toolbox of MATLAB.

2 Problem statement

Consider a continuous-time uncertain system of the form

$$\dot{\boldsymbol{x}}(t) = (A_0 + \Delta A(t))\boldsymbol{x}(t) + B_0\boldsymbol{u}(t) \tag{1}$$
$$\boldsymbol{y}(t) = (C_0 + \Delta C(t))\boldsymbol{x}(t) \tag{2}$$

where $\boldsymbol{x}(t) \in \Re^n$ is the state vector, $\boldsymbol{u}(t) \in \Re^m$ is the control input vector, $\boldsymbol{y}(t) \in \Re^p$ is the measured output, A_0, B_0, C_0 are known constant real-valued matrices of appropriate dimensions, $\Delta A(t)$, $\Delta C(t)$ denote real-valued matrix functions representing parameter uncertainties. It is assumed that the system uncertainties have the form

$$\Delta A(t) = D_1 F_1(t) E_1$$
$$\Delta C(t) = D_2 F_2(t) E_2$$

with

$$F_i^T(t) F_i(t) \leq I$$

and where D_1, D_2, E_1, E_2 are known constant real-valued matrices of appropriate dimensions and $F_1(t)$ and $F_2(t)$ are unkonwn real time-varying matrices

The problem considered here is to design an observer-based controller of the form

$$\dot{\hat{\boldsymbol{x}}}(t) = A_0\hat{\boldsymbol{x}}(t) + B_0\boldsymbol{u}(t) + K_o(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)) \tag{3}$$
$$\hat{\boldsymbol{y}}(t) = C_0\hat{\boldsymbol{x}}(t) \tag{4}$$
$$\boldsymbol{u}(t) = K_c\hat{\boldsymbol{x}}(t) \tag{5}$$

which gives an upper bound on the following quadratic performance index associated with the uncertain system (1) and (2)

$$J = \int_0^\infty (\boldsymbol{x}^T(t) Q \boldsymbol{x}(t) + \boldsymbol{u}^T(t) R \boldsymbol{u}(t)) dt \tag{6}$$

where Q and R are given positive-definite symmetric matrices.

3 Main results

In this section, a sufficient condition is established for the existence of an observer-based guaranteed cost controller for the uncertain system (1) and (2). Here, it is assumed that

$$K_c = -R^{-1}B_0^T P \tag{7}$$

Theorem 1. The feedback control law (3)-(5) with (7) is an observer-based guaranteed cost controller if there exist a matrix $K_o \in \Re^{m\times n}$, a symmetric positive-definite matricx $P \in \Re^{n\times n}$, such that the following matrix inequality holds

$$\Omega < 0 \tag{8}$$

where

$$\begin{aligned}
\Omega &= \begin{bmatrix} \Sigma_1 & (\Delta A - K_o\Delta C)^T P \\ P(\Delta A - K_o\Delta C) & \Sigma_2 \end{bmatrix} \\
\Sigma_1 &= P(A_0 + \Delta A) + (A_0 + \Delta A)^T P \\
&\quad + Q - PB_0R^{-1}B_0^T P \\
\Sigma_2 &= P(A_0 - K_oC_0) + (A_0 - K_oC_0)^T P \\
&\quad + PB_0R^{-1}B_0^T P
\end{aligned}$$

Moreover, the performance index is evaluated as

$$J < \boldsymbol{\phi}^T(0)P\boldsymbol{\phi}(0) + \boldsymbol{\psi}^T(0)P\boldsymbol{\psi}(0) \tag{9}$$

Proof. The input (3)-(5) with (7) yields the closed-loop system

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (A_0 + \Delta A(t) - B_0R^{-1}B_0^T P)\boldsymbol{x}(t) \\
&\quad + B_0R^{-1}B_0^T P\boldsymbol{e}(t) \qquad (10) \\
\dot{\boldsymbol{e}}(t) &= (A_0 - K_oC_0)\boldsymbol{e}(t) \\
&\quad + (\Delta A(t) - K_o\Delta C(t))\boldsymbol{x}(t) \qquad (11)
\end{aligned}$$

where $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ is the estimated error of the system. Define a candidate of Lyapunov function as

$$V(\boldsymbol{x},\boldsymbol{e}) = \boldsymbol{x}^T(t)P\boldsymbol{x}(t) + \boldsymbol{e}^T(t)P\boldsymbol{e}(t) \tag{12}$$

then, the time derivative of (12) along to (10) and (11) is calculated as

$$\begin{aligned}
\dot{V}(\boldsymbol{x},\boldsymbol{e}) &= 2\boldsymbol{x}^T(t)P\dot{\boldsymbol{x}}(t) + 2\boldsymbol{e}^T(t)P\dot{\boldsymbol{e}}(t) \\
&= 2\boldsymbol{x}^T(t)P\{(A_0 + \Delta A(t) - B_0R^{-1}B_0^T P)\boldsymbol{x}(t) \\
&\quad + B_0R^{-1}B_0^T P\boldsymbol{e}(t)\} \\
&\quad + 2\boldsymbol{e}^T(t)P\{(A_0 - K_oC_0)\boldsymbol{e}(t) \\
&\quad + (\Delta A(t) - K_o\Delta C(t))\boldsymbol{x}(t)\} \\
&= \boldsymbol{z}^T(t)\Omega\boldsymbol{z}(t) \\
&\quad - (\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)) \qquad (13)
\end{aligned}$$

where

$$\boldsymbol{z}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{e}(t) \end{bmatrix} \tag{14}$$

Applying (8) to (13) gives

$$\dot{V}(\boldsymbol{x},\boldsymbol{e}) < -(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)) < 0 \tag{15}$$

for any $\boldsymbol{x}(t) \neq \mathbf{0}$. Thus, the closed-loop system is asymptotically stable.

Further, integrating (15) from 0 to T leads to

$$\begin{aligned}
&\boldsymbol{x}^T(T)P\boldsymbol{x}(T) - \boldsymbol{x}^T(0)P\boldsymbol{x}(0) \\
&+ \boldsymbol{e}^T(T)P\boldsymbol{e}(T) - \boldsymbol{e}^T(0)P\boldsymbol{e}(0) \\
<\ & -(\boldsymbol{x}^T(t)Q\boldsymbol{x}(t) + \boldsymbol{u}^T(t)R\boldsymbol{u}(t)) < 0 \qquad (16)
\end{aligned}$$

Here, the aymptotic stability of the closed-loop system implies that

$$\boldsymbol{x}^T(T)P\boldsymbol{x}(T) \to 0, \quad \boldsymbol{e}^T(T)P\boldsymbol{e}(T) \to 0 \tag{17}$$

as T tends to the infinity. Hence, it is obtained that

$$\begin{aligned}
J &= \int_0^\infty (\boldsymbol{x}^T(\tau)Q\boldsymbol{x}(\tau) + \boldsymbol{u}^T(\tau)R\boldsymbol{u}(\tau))d\tau \\
&< \boldsymbol{x}^T(0)P\boldsymbol{x}(0) + \boldsymbol{x}^T(0)P\boldsymbol{x}(0) \\
&= \boldsymbol{\phi}^T(0)P\boldsymbol{\phi}(0) + \boldsymbol{\psi}^T(0)P\boldsymbol{\psi}(0) \qquad (18)
\end{aligned}$$

As a result, the proof is complete. □

Next, on the basis of Theorem 1, we prove another sufficient condition without uncertain parameters. Before stating Theorem 2, a necessary lemma will be introduced.

Lemma 1 [5]. Let D and E be matrices of appropriate dimensions, and F be a matrix function satisfying $F^TF \leq I$. Then for any positive scalar α, the following inequality holds

$$DFE + E^TF^TD^T \leq \alpha DD^T + \alpha^{-1}E^TE \tag{19}$$

Theorem 2. If there exist scalars $\gamma > 0$, $\delta > 0$, $\epsilon > 0$, a matrix K_o, a symmetric positive-definite matrix P such that the following matrix inequality

$$\begin{bmatrix} \Sigma_3 & 0 \\ 0 & \Sigma_4 \end{bmatrix} < 0 \tag{20}$$

where

$$\begin{aligned}
\Sigma_3 &= PA_0 + A_0^T P - PB_0R^{-1}B_0^T P + Q \\
&\quad + \gamma PD_1D_1^T P + \frac{1}{\gamma}E_1^TE_1 + \frac{1}{\epsilon}E_1^TE_1 + \frac{1}{\delta}E_2^TE_2 \\
\Sigma_4 &= P(A_0 - K_oC_0) + (A_0 - K_oC_0)^T P \\
&\quad + PB_0R^{-1}B_0^T P + \epsilon PD_1D_1^T P \\
&\quad + \delta PK_oD_2D_2^TK_o^T P
\end{aligned}$$

is satisfied, then the memoryless state feedback control law (5) is an observer-based guaranteed cost controller and

$$J^* = \boldsymbol{\phi}^T(0)P\boldsymbol{\phi}(0) + \boldsymbol{\psi}^T(0)P\boldsymbol{\psi}(0) \tag{21}$$

is a guaranteed cost for the uncertain system (1) and (2).

Proof. By applying Lemma 1, it follows for any $\gamma > 0,\ \delta > 0,\ \epsilon > 0$ that

$$\begin{aligned}
&2\boldsymbol{x}^T(t)P\Delta A\boldsymbol{x}(t) = 2\boldsymbol{x}^T(t)PD_1F_1E_1\boldsymbol{x}(t)\\
&\quad\le \gamma\boldsymbol{x}^T(t)PD_1D_1^TP\boldsymbol{x}(t) + \frac{1}{\gamma}\boldsymbol{x}^T(t)E_1^TE_1\boldsymbol{x}(t) \qquad (22)\\
&2\boldsymbol{e}^T(t)P\Delta A\boldsymbol{x}(t) = 2\boldsymbol{e}^T(t)PD_1F_1E_1\boldsymbol{x}(t)\\
&\quad\le \epsilon\boldsymbol{e}^T(t)PD_1D_1^TP\boldsymbol{e}(t) + \frac{1}{\epsilon}\boldsymbol{x}^T(t)E_1^TE_1\boldsymbol{x}(t) \qquad (23)\\
&-2\boldsymbol{e}^T(t)PK_o\Delta C\boldsymbol{x}(t) = -2\boldsymbol{e}^T(t)PK_oD_2F_2E_2\boldsymbol{x}(t)\\
&\quad\le \delta\boldsymbol{e}^T(t)PK_oD_2D_2^TK_o^TP\boldsymbol{e}(t)\\
&\qquad + \frac{1}{\delta}\boldsymbol{x}^T(t)E_2^TE_2\boldsymbol{x}(t) \qquad (24)
\end{aligned}$$

Substituting (22), (23), (24) into (8) yields the desired result. □

Theorem 3. For a given pair of $\delta > 0,\ \epsilon > 0$, if the following LMI optimization problem; $\min\{\mathrm{tr}(P)\}$

$$\begin{bmatrix}
\Sigma_5 & XE_1^T & XE_1^T & XE_2^T & X\\
E_1X & -\gamma I & 0 & 0 & 0\\
E_1X & 0 & -\epsilon I & 0 & 0\\
E_2X & 0 & 0 & -\delta I & 0\\
X & 0 & 0 & 0 & -Q^{-1}
\end{bmatrix} < 0 \tag{25}$$

where

$$\Sigma_5 = A_0X + XA_0^T - B_0R^{-1}B_0^T + \gamma D_1D_1^T$$

has a solution of scalar $\gamma > 0$, and symmetric positive-definite matrix X, and if

$$\begin{bmatrix}
\Sigma_6 & PK_oD_2 & PB_0 & PD_1\\
D_2^TK_o^TP & -\frac{1}{\delta}I & 0 & 0\\
B_0^TP & 0 & -R & 0\\
D_1^TP & 0 & 0 & -\frac{1}{\epsilon}I
\end{bmatrix} < 0 \tag{26}$$

where

$$\Sigma_6 = PA_0 + A_0^TP - PK_oC_0 - C_0^TK_o^TP$$

has a solution of matrix K_o, then the control law

$$\boldsymbol{u}(t) = -R^{-1}B_0^TX^{-1}\hat{\boldsymbol{x}}(t) \tag{27}$$

is a suboptimal guaranteed cost controller which gives the optimal value of the guaranteed cost (21) for the given parameters $\delta > 0,\ \epsilon > 0$.

Proof. Pre- and post-multiplying Σ_3 by P^{-1} on both sides, and denoting $X = P^{-1}$ lead to the equivalent inequality

$$\begin{aligned}
&A_0X + XA_0^T - B_0R^{-1}B_0^T + \gamma D_1D_1^T + \frac{1}{\gamma}XE_1^TE_1X\\
&+\frac{1}{\epsilon}XE_1^TE_1X + \frac{1}{\delta}XE_2^TE_2X + XQX < 0 \qquad (28)
\end{aligned}$$

It follows from Schur Complement that (28) is equivalent to (25).

Next, using Schur Complement for Σ_4, we obtain

$$\begin{bmatrix}
\Sigma_6 & PK_oD_2 & PB_0 & PD_1\\
D_2^TK_o^TP & -\frac{1}{\delta}I & 0 & 0\\
B_0^TP & 0 & -R & 0\\
D_1^TP & 0 & 0 & -\frac{1}{\epsilon}I
\end{bmatrix} < 0 \tag{29}$$

$$\Sigma_6 = PA_0 + A_0^TP - PK_oC_0 - C_0^TK_o^TP$$

For given scalars $\delta > 0,\ \epsilon > 0$, if there exist $\gamma,\ X$ in (25) which is a solution of optimal LMI problem; $\min\{\mathrm{tr}(P)\}$, and if there exists a matrix K_o which satisfies (26) using $P = X^{-1}$, the guaranteed cost under $\delta,\ \epsilon$ is suboptimal. □

Remark 1. The suboptimal guaranteed cost controller (27) for over all parameters δ, ϵ can be determined by a search such as the optimization problem in Theorem 3 has a solution.

4 An illustrative example

Consider the uncertain time-delay system described by the state equation

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= (A_0 + \Delta A(t))\boldsymbol{x}(t) + B_0\boldsymbol{u}(t) && (30)\\
\boldsymbol{y}(t) &= (C_0 + \Delta C(t))\boldsymbol{x}(t) && (31)
\end{aligned}$$

and full state observer for nominal part of the system

$$\begin{aligned}
\dot{\hat{\boldsymbol{x}}}(t) &= A_0\hat{\boldsymbol{x}}(t) + B_0\boldsymbol{u}(t) + K_o(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)) && (32)\\
\hat{\boldsymbol{y}}(t) &= C_0\hat{\boldsymbol{x}}(t) && (33)
\end{aligned}$$

and the performance index (6), where

$$A_0 = \begin{bmatrix} 0 & 1\\ -1 & 1\end{bmatrix},\quad B_0 = \begin{bmatrix} 0.2\\ 3\end{bmatrix},$$

$$C_0 = \begin{bmatrix} 1 & 2\end{bmatrix},\quad D_1 = \begin{bmatrix} 0.1 & 0.1\\ 0 & 0\end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.3 & 0.1\end{bmatrix},\quad E_1 = E_2 = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}$$

and $\delta = 4.3, \quad \epsilon = 8.8$

The suboptimal observer-based guaranteed cost controller can be determined by solving (25) and (26). We obtain

$$P = \begin{bmatrix} 2.5925 & 0.2558 \\ 0.2558 & 0.5849 \end{bmatrix}, \quad K_o = \begin{bmatrix} 0.1142 \\ 7.9563 \end{bmatrix},$$
$$K_c = -R^{-1}B_0^T P = \begin{bmatrix} -1.2858 & -1.8057 \end{bmatrix},$$
$$\gamma = 4.3974, \quad J^* = 6.3547$$

The simulation results are shown in Figs. 1-4.

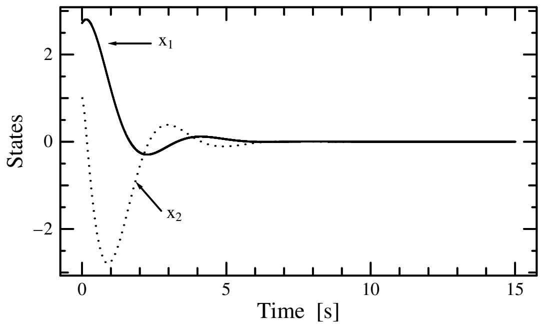

Fig.1 Trajectories of states.

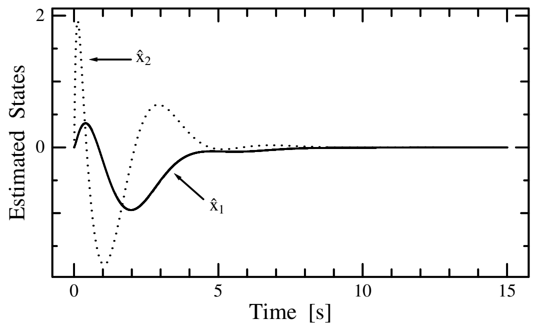

Fig.2 Trajectories of estimated states.

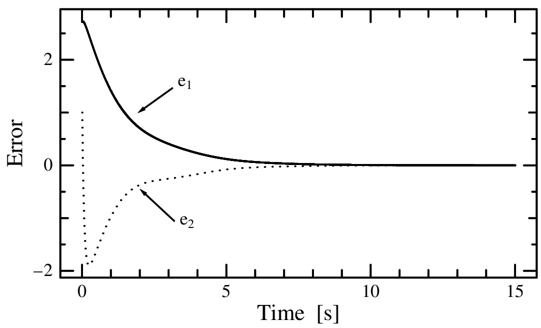

Fig.3 Trajectories of errors.

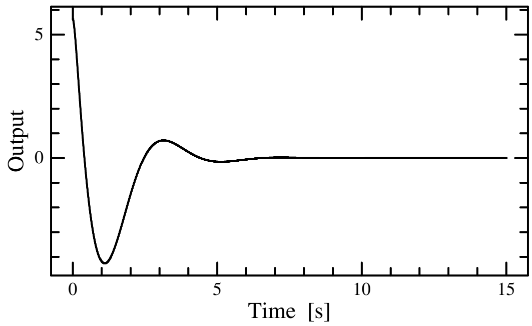

Fig.4 Trajectory of output.

5 Summary

This paper discusses an LMI approach to observer-based guaranteed cost control problem. A sufficient condition for the existence of memoryless state feedback guaranteed cost controllers is derived on the basis of the LMI approach. A numerical example shows the potential of the proposed method.

References

[1] S. S. L. Chang and T. K. C. Peng: Adaptive guaranteed cost control of systems with uncertain parameters," *IEEE Transactions on Automatic Control*, **AC-17,** (4), pp. 474-483, 1972

[2] C. H. Lien: Robust observer-based control of systems with state perturbations via LMI approach," *IEEE Transactions on Automatic Control*, **AC-49,** (8), pp.1365-1370, 2004

[3] S. Won, and J. H. Park: Design of observer-based controller for perturbed time-delay systems," *JSME International Journal C*, **42,** (1), pp.129-132, 1999

[4] C. H. Lien: Guaranteed cost observer-based controls for a class of uncertain neutral time-delay systems," *Journalof Optimization Theory and Applications*, **126,** (1), pp.137-156, 2005

[5] M. S. Mahmoud, and M. Zribi: Guaranteed cost observer-based control of uncertain time-lag systems," *Computers and Electrical Engineering*, **29,** pp.193-212, 2003