Guaranteed cost control of discrete time system with performance index including cross term

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1 Introduction

Guaranteed cost control problem is the design method of the robust control system [1]. This method guarantees the robust stability of disturbed control system by using existing of the upper bound of quadratic performance index. Kono extended this method to the case with cross term in performance index and show the condition for robust stability [2]. In this paper, we extend this problem to the discrete time system. As the result of formulation, stochastic discrete Riccati equation is obtained. We show and prove the condition of the closed loop system to be stable. Finally, through showing the numerical example, we validate out method.

2 Derivation of discrete time stochastic Riccati equation

In this section, we derive the stochastic discrete Riccati equation with performance index including cross term of input vector and state vector. Wonham proposed this equation which is obtained as the result of stochastic control problem [3]. In this literature, the stochastic discrete Riccati equation is formed as the discrete Riccati equation with additional structured term. In this section, we extend this problem to the performance index with cross term with state vector and input vector. We abbreviate stochastic discrete Riccati equation as SDARE.

Let us consider the discrete time linear control system with uncertainty in state matrix,

$$x(k+1) = A(\xi)x(k) + Bu(k) \tag{1}$$

$A(\xi)$ is defined as

$$A(\xi) = A_0 + \sum_{i=1}^{p} \xi_i A_i \tag{2}$$

where A_0 is nominal structure of the state matrix, A_i is structure of the uncertainty, and ξ_i is size of the uncertainty. Performance index is defined as

$$\begin{aligned} J &= \sum_{k=0}^{\infty} \left\{ x(k)^T Q x(k) + u(k)^T R u(k) + 2x(k)^T S u(k) \right\} \\ &= \sum_{k=0}^{\infty} l(x,u) \end{aligned} \tag{3}$$

where $Q \geq 0, R > 0$ and $S \geq 0$ are weighting matrices for state vector, input vector and cross term of state and input vector, respectively. These matrices take appropriate dimension size. Lyapunov function $V(\cdot)$ is

$$V(x(k)) = x(k)^T P(k) x(k) \tag{4}$$

From the principle of optimality, we have

$$\begin{aligned} & H(V, x, u, \xi) \\ = \ & l(x,u) + V(x(k+1), k+1) - V(x(k), k) \\ = \ & x^T C^T C x + u^T R u + 2x^T S u - x^T P(k) x \\ & +(A(\xi)x + Bu)^T P(k+1)(A(\xi)x + Bu) \end{aligned}$$

$$
\begin{aligned}
&= x^T C^T C x + u^T R u + 2x^T S u \\
&\quad + x^T A(\xi)^T P(k+1) A(\xi) x + u^T B^T P(k+1) B u \\
&\quad + 2x^T A(\xi)^T P(k+1) B u - x^T P(k) x \le 0
\end{aligned} \tag{5}
$$

In the LQR problem of discrete time system, by using solution $P(k)$ of the SDARE, the optimal input vector $u(k)$ is defined as follows,

$$
u(k) = -(B^T P(k+1) B + R)^{-1} (A_0^T P(k+1) B + S)^T x(k) \tag{6}
$$

Then, let $(B^T P B + R)^{-1}(A_0^T P B + S)^T = \Omega(P)$ and substitute into equation (5), we have

$$
\begin{aligned}
&= x^T C^T C x + x^T \Omega(P(k+1))^T R \Omega(P(k+1)) x \\
&\quad - x^T S \Omega(P(k+1)) x - x^T \Omega(P(k+1))^T S^T x \\
&\quad + x^T A(\xi)^T P(k+1) A(\xi) x \\
&\quad + x^T \Omega(P(k+1))^T B^T P(k+1) B \Omega(P(k+1)) x \\
&\quad - x^T A(\xi)^T P(k+1) B \Omega(P(k+1)) x \\
&\quad - x^T \Omega(P(k+1))^T B^T P(k+1) A(\xi) x - x^T P(k) x \\
&\quad \le 0
\end{aligned}
$$

Now, let us substitute structured uncertainty $A(\xi) = A_0 + \Delta A$, then

$$
\begin{aligned}
&= x^T C^T C x + x^T \Omega(P(k+1))^T R \Omega(P(k+1)) x \\
&\quad - x^T S \Omega(P(k+1)) x - x^T \Omega(P(k+1))^T S^T x \\
&\quad + x^T A_0^T P(k+1) A_0 x + x^T \Delta A^T P(k+1) \Delta A x \\
&\quad + x^T A_0^T P(k+1) \Delta A x + x^T \Delta A^T P(k+1) A_0 x \\
&\quad + x^T \Omega(P(k+1))^T B^T P(k+1) B \Omega(P(k+1)) x \\
&\quad - x^T A_0^T P(k+1) B \Omega(P(k+1)) x \\
&\quad - x^T \Omega(P(k+1))^T B^T P(k+1) A_0 x \\
&\quad - x^T \Delta A^T P(k+1) B \Omega(P(k+1)) x \\
&\quad - x^T \Omega(P(k+1))^T B^T P(k+1) \Delta A x - x^T P(k) x \\
&= x^T \big\{ C^T C + A_0^T P(k+1) A_0 + \Omega(P(k+1))^T B^T \\
&\quad \cdot P(k+1) B \Omega(P(k+1)) - A_0^T P(k+1) B \\
&\quad \cdot \Omega(P(k+1)) - \Omega(P(k+1))^T B^T P(k+1) A_0 \\
&\quad + \Omega(P(k+1))^T R \Omega(P(k+1)) - S \Omega(P(k+1)) \\
&\quad - \Omega(P(k+1))^T S^T - P(k) + \Delta A^T P(k+1) \Delta A \\
&\quad + A_0^T P(k+1) \Delta A + \Delta A^T P(k+1) A_0 \\
&\quad - \Delta A^T P(k+1) B \Omega(P(k+1)) \\
&\quad - \Omega(P(k+1))^T B^T P(k+1) \Delta A \big\} x \le 0
\end{aligned} \tag{7}
$$

From the positive semi-definitively of $H(\cdot)$, we obtain following inequality

$$
\begin{aligned}
T(x,k,P) &= T_0(P(k+1)) + T_1(P(k+1)) \\
&\quad + C^T C - P(k) \le 0
\end{aligned} \tag{8}
$$

Now, $T_0(\cdot)$ and $T_1(\cdot)$ are

$$
\begin{aligned}
&T_0(P(k+1)) \\
&= A_0^T P(k+1) A_0 - (A_0^T P(k+1) B + S)(B^T \\
&\quad \cdot P(k+1) B + R)^{-1} (A_0^T P(k+1) B + S)^T
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
&T_1(P(k+1)) \\
&= \Delta A^T P(k+1) \Delta A + \Delta A^T P(k+1) A_0 \\
&\quad + A_0^T P(k+1) \Delta A - \Delta A^T P(k+1) B \Omega(P(k+1)) \\
&\quad - \Omega(P(k+1))^T B^T P(k+1) \Delta A
\end{aligned} \tag{10}
$$

where, let U_1 is the upper bound matrix of $T_1(\cdot)$, form inequality condition (8), we have following difference equation.

$$
T_0(P(k+1)) + C^T C + U_1 - P(k) = 0 \tag{11}
$$

Suppose that there exists stationary solution of equation (11)

$$
P = T_0(P) + C^T C + U_1 \tag{12}
$$

Therefore, we obtain the SDARE with performance index including cross term as follows

$$
\begin{aligned}
P &= A_0^T P A_0 - (A_0^T P B + S)(B^T P B + R)^{-1} \\
&\quad \cdot (A_0^T P B + S)^T + C^T C + U_1
\end{aligned} \tag{13}
$$

In equation (13), if we omit the upper bound matrix, it coincide with the SDARE of nominal system.

3 Robust stability

In this section, under the assumption that there exists a solution P of SDARE, we prove stability of the closed-loop system which is designed by using out proposed method

Theorem 1 *In the system (1), optimal input u^* which minimized the performance index (3) is obtained*

$$\begin{aligned} u^*(k) &= -(B^TPB+R)^{-1}(A_0^TPB+S)x(k) \\ &= -\Omega(P)x(k) \end{aligned} \quad (14)$$

Then the closed-loop system $A_c(\cdot)$ is

$$x(k+1) = A_c(A(\xi), B, P)x(t) \quad (15)$$

where

$$A_c(A(\xi), B, \Omega(P)) = A(\xi) - B\Omega(P)$$

Now, we suppose following assumption

Assumption 1 *Let*

$$\begin{aligned} &A_c^TPB\Omega(P) + \Omega(P)^TB^TPA_c \\ &\quad -\Omega(P)^TR\Omega(P) + C^TC = D^TD, \end{aligned} \quad (16)$$

then D^TD is positive semi-definite.

Next, we derive theorem for asymptotical stability of closed-loop system.

Theorem 2 *The closed-loop system (15) with optimal feedback control input (14) is asymptotic stability, if there exists positive semi-definite solution P of SDARE (13) and assumption 1 is satisfied.*

(proof of Theorem 2) In discrete time system, from stability condition in the sense of Lyapunov, we have

$$A_c^TPA_c - P \leq 0 \quad (17)$$

Substitute A_c into inequality (17), left-hand side of the inequality becomes

$$\begin{aligned} &= (A(\xi) - B\Omega(P))^TP(A(\xi) - B\Omega(P)) - P \\ &= A(\xi)^TPA(\xi) - A(\xi)^TPB\Omega(P) - \Omega(P)^TB^TPA(\xi) \\ &\quad +\Omega(P)^TB^TPB\Omega(P) - P \end{aligned} \quad (18)$$

Substitute $A(\xi) = A_0 + \Delta A$

$$\begin{aligned} &= A_0^TPA_0 + A_0^TP\Delta A + \Delta A^TPA_0 + \Delta A^TP\Delta A \\ &\quad -A_0^TPB\Omega(P) - \Delta A^TPB\Omega(P) \\ &\quad -\Omega(P)^TB^TPA_0 - \Omega(P)^TB^TP\Delta A \\ &\quad +\Omega(P)^TB^TPB\Omega(P) - P \\ &= A_0^TPA_0 - A_0^TPB\Omega(P) - \Omega(P)^TB^TPA_0 \\ &\quad +\Omega(P)^TB^TPB\Omega(P) - P + T_1(P) \end{aligned} \quad (19)$$

Where, we substitute P of equation (13) into (19), we have

$$\begin{aligned} &= A_0^TPA_0 - A_0^TPB\Omega(P) - \Omega(P)^TB^TPA_0 \\ &\quad +\Omega(P)^TB^TPB\Omega(P) - P + T_1(P) \\ &= A_0^TPA_0 - A_0^TPB\Omega(P) - \Omega(P)^TB^TPA_0 \\ &\quad +\Omega(P)^TB^TPB\Omega(P) + T_1(P) \\ &\quad -\{A_0^TPA_0 - \Omega(P)^T(B^TPB+R)\Omega(P) \\ &\quad +C^TC + U_1\} \\ &= -A_0^TPB\Omega(P) + \Omega(P)^TB^TPB\Omega(P) \\ &\quad -\Omega(P)^TB^TPA_0 + \Omega(P)^TB^TPB\Omega(P) \\ &\quad +\Omega(P)^TR\Omega(P) - C^TC - (U_1 - T_1(P)) \\ &= -(A_0 - B\Omega(P))^TPB\Omega(P) - \Omega(P)^TB^TP \\ &\quad \cdot(A_0 - B\Omega(P)) + \Omega(P)^TR\Omega(P) - C^TC \\ &\quad -(U_1 - T_1(P)) \\ &= -\left\{A_c^TPB\Omega(P) + \Omega(P)^TB^TPA_c - \Omega(P)^TR\Omega(P) \right. \\ &\quad \left. +C^TC\right\} - (U_1 - T_1(P)) \leq 0 \end{aligned} \quad (20)$$

From equation (20), we obtain the stabilizable condition

$$\begin{aligned} &A_c^TPB\Omega(P) + \Omega(P)^TB^TPA_c \\ &-\Omega(P)^TR\Omega(P) + C^TC \geq 0 \end{aligned} \quad (21)$$

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4 Eigenvalue upper bound

From equation (10), uncertainty is described as

$$\begin{aligned} T_1(P) &= \Delta^TAP\Delta A + \Delta A^TP(A_0 - B\Omega(P)) \\ &\quad +(A_0 - B\Omega(P))^TP\Delta A \end{aligned} \quad (22)$$

where P is the stationary solution of SDARE. From equation (2), substitute uncertainty of system matrix $\Delta A = \sum_{i=1}^{p} \xi_i A_i$ and we obtain

$$\begin{aligned} = \quad & \sum_{i=1}^{p} \xi_i \zeta_j D_{ij} + \sum_{i=1}^{p} \left[A_i^T P (A_0 - B\Omega(P)) \right. \\ & \left. + (A_0 - B\Omega(P))^T P A_i \right] \end{aligned} \tag{23}$$

where

$$D_{ij} = A_i^T P A_j + A_j^T P A_i$$

Because $A_i^T P(A_0 - B\Omega(P)) + (A_0 - B\Omega(P))^T P A_i$ and D_{ij} are symmetric matrices, then there exist orthogonal matrices Y_i and Z_{ij} which satisfy

$$\begin{aligned} & Y_i^T \left[A_i^T P (A_0 - B\Omega(P)) \right. \\ & \qquad \left. + (A_0 - B\Omega(P))^T P A_i \right] Y_i = \Lambda_i \end{aligned} \tag{24}$$

$$Z_{ij}^T D_{ij} Z_{ij} = \Gamma_{ij} \tag{25}$$

where Λ_i and Γ_{ij} are diagonal matrices. By using Y_i, D_{ij}, Λ_i and Γ_{ij}, the upper bound matrix of $T(P)$ is expressed as

$$U_E = \sum_{i=1}^{p} Y_i^T \|\Lambda_i\| Y_i + \frac{1}{2} \sum_{i=1}^{p} \sum_{j=1}^{p} Z_{ij}^T \|\Gamma_{ij}\| Z_{ij} \tag{26}$$

where U_E is called eigenvalue upper bound matrix. In the next section, we show the numerical example.

5 Numerical example

We consider following system parameter,

$$A_0 = \begin{bmatrix} 0 & 1 \\ 1.5 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$S = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad R = 1$$

Using these parametr, we solve the SDARE with eigenvalue upper bound and obtain stationary solution P. Poles of the nominal closed-loop system is obtained as

$$(\quad 0.40825 \quad , \quad -0.40825 \quad)$$

Poles of the perturbed closed-loop system is obtained as

$$(\quad 0.81650 \quad , \quad -0.81650 \quad)$$

This result shows that the perturbed system remains in stable, then we had confirm robust stability for disturbance of our proposed method.

6 Conclution

In this paper, we consider the guaranteed cost control problem for the performance index including cross term of discrete time system. We show the structure of uncertainty and discuss about stationary condition. We apply eigenvalue upper bound matrix to this problem and show the numerical example. From this result, we confirm the robust stability of the closed-loop system for the system matrix disturbance. Future study is to consider about linear upper bound. It is pointed out that there exists relationship between linear upper bound matrix and LMI solution of the structured uncertain system.

References

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