Remarks on Recognizability of Topological Components by Three-Dimensional Automata

Yuusuke NAGAMIZU[1], Makoto SAKAMOTO[1], Takao ITO[1], Kouichi IIHOSHI[1], Naoko TOMOZOE[1], Hiroshi FURUTANI[1], Michio KONO[1], Satoshi IKEDA[1], Tatsuhiro TAMAKI[2], and Katsushi INOUE[3]

[1]Dept. of Computer Science and Systems Engineering, University of Miyazaki, Miyazaki 889-2192, JAPAN

[2]Dept. of Business Administration, Ube National College of Technology, Ube 755-8555, JAPAN

[3]Dept. of Computer Science and Systems Engineering, Yamaguchi University, Ube 755-8611, JAPAN

Abstract

It is conjectured that the three-dimensional pattern processing has its our difficulties not arising in two-dimensional case. One of these difficulties occurs in recognizing topological properties of three-dimensional patterns because the three-dimensional neighborhood is more complicated than two-dimensional case. Generally speaking, a property or relationship is topological only if it is preserved when an arbitrary ' rubber-sheet ' distortion is applied to the pictures . For example, adjacency and connectedness are topological ; area, elongatedness, convexity, straightness, etc. are not. In recent years, there have been many interesting papers on digital topological properties. For example, an interlocking component was defined as a new topological property in three-dimensional digital pictures, and it was proved that no one marker automaton can recognize interlocking components in a three-dimensional digital picture. In this paper, we deal with recognizability of topological components by three-dimensional Turing machines, and investigate some properties.

KeyWords : digital geometry,interlocking component, one marker automaton, three-dimensional automaton, Turing machine, topological component

1 Introduction

Digital geometry has played an important role in computer image analysis and recognition[3]. In particular, there is a well-developed theory of *topological* properties such as *connectedness* and *holes* for two-dimensional arrays[4]. On the other hand, three-dimensional information processing has also become of increasing interest with the rapid growth of computed tomography, robotics, and so on. Thus it has become desirable to study the geometrical properties such as *interlocking components* and *cavities* for three-dimensional arrays[2,5]. In[2], interlocking components was proposed as a new topological property of three-dimensional digital pictures : Let S_1 and S_2 be two subsets of the same three-dimensional digital picture. S_1 and S_2 are said to be interlocked when they satisfy the following conditions:

(1) S_1 and S_2 are *toruses*,
(2) S_1 goes through a *hole* of S_2,
(3) S_2 goes through a *hole* of S_1.

The interlocking of S_1 and S_2 is illustrated in Fig.1. This relation may be considered as a chainlike connectivity.

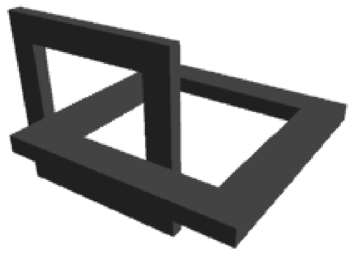

Fig. 1: Interlocking components.

It is proved that no one marker automaton can recognize interlocking components in a three-dimensional digital picture in [2]. In this paper, we investigate recognizability of topological properties such as interlocking components by three-dimensional Turing machines.

2 Preliminaries

Definition 2.1. Let Σ be a finite set of symbols. A *three-dimensional tape* over Σ is a three-dimensional rectangular array of elements of Σ. The set of all three-dimensional tapes over Σ is denoted by $\Sigma^{(3)}$. Given a tape $x \in \Sigma^{(3)}$, for each $j(1 \leq j \leq 3)$, we let $l_j(x)$ be the length of x along the j^{th} axis. When $1 \leq i_j \leq l_j(x)$ for each $j(1 \leq j \leq 3)$, let $x(i_1, i_2, i_3)$ denote the symbol in x with coordinates (i_1, i_2, i_3), as shown in Fig. 2. Furthermore, we define

$$x[(i_1, i_2, i_3), (i'_1, i'_2, i'_3)],$$

when $1 \leq i_j \leq i'_j \leq l_j(x)$ for each integer $j(1 \leq j \leq 3)$, as the three-dimensional tape y satisfying the following :

(i) for each $j(1 \leq j \leq 3)$, $l_j(y) = i'_j - i_j + 1$;

(ii) for each r_1, r_2, r_3 $(1 \leq r_1 \leq l_1(y)$, $1 \leq r_2 \leq l_2(y)$, $1 \leq r_3 \leq l_3(y)$, $y(r_1, r_2, r_3) = x(r_1 + i_1 - 1, r_2 + i_2 - 1, r_3 + i_3 - 1)$.

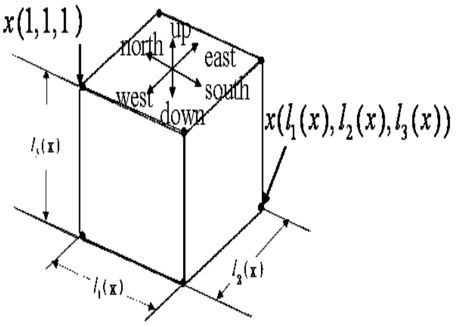

Fig. 2: Three-dimensional input tape.

Definition 2.2. *A three-dimensional nondeterministic one-marker automaton* 3-NM_1 is defined by the six-tuple

$$M = (Q, q_0, F, \Sigma, \{+, -\}, \delta),$$

where

(1) Q is a finite set of *states*;

(2) $q_0 \in Q$ is the *initial state*;

(3) $F \subseteq Q$ is the set of *accepting states*;

(4) Σ is a finite input alphabet ($\sharp \notin \Sigma$ is the *boundary symbol*);

(5) $\{+,-\}$ is the pair of signs of presence and absence of the marker; and

(6) δ: $(Q \times \{+,-\}) \times ((\Sigma \cup \{\sharp\}) \times \{+,-\}) \rightarrow 2^{(Q \times \{+,-\})} \times ((\Sigma \cup \{\sharp\}) \times \{+,-\}) \times$ {east,west,south,north,up,down,no move}) is the *next-move function*, satisfying the following: For any $q, q' \in Q$, any $a, a' \in \Sigma$, any $u, u', v, v' \in \{+,-\}$, and any $d \in$ {east,west,south,north,up,down,no move}, if $((q',u'),(a',v'),d) \in \delta((q,u),(a,v))$ then $a = a'$, and $(u,v,u',v') \in \{(+,-,+,-),(+,-,-,+),(-,+,-,+),(-,+,+,-),(-,-,-,-)\}$.

We call a pair (q,u) in $Q \times \{+,-\}$ an *extended state*, representing the situation that M holds or does not hold the marker in the finite control according to the sign $u = +$ or $u = -$, respectively. A pair (a,v) in $\Sigma \times \{+,-\}$ represents an input tape cell on which the marker exists or does not exsit according to the sign $v = +$ or $v = -$, respectively.

Therefore, the restrictions on δ above imply the following conditions. (A) When holding the marker, M can put it down or keep on holding. (B) When not holding the marker, and (i) if the marker exists on the current cell, M can pick it up or leave it there, or (ii) if the marker does not exist on the current cell, M cannot create a new marker any more.

Definition 2.3. Let Σ be the input alphabet of 3-NM_1 M. An *extended input tape* $\tilde{x}$ of M is any three-dimensional tape over $\Sigma \times \{+,-\}$ such that

(i) for each $j(1 \leq j \leq 3)$, $l_j(\tilde{x}) = l_j(x)$,

(ii) for each $i_1(1 \leq i_1 \leq l_1(\tilde{x}))$, $i_2(1 \leq i_2 \leq l_2(\tilde{x}))$, and $i_3(1 \leq i_3 \leq l_3(\tilde{x}))$, $\tilde{x}(i_1, i_2, i_3) = x(i_1, i_2, i_3, u)$ for some $u \in \{+, -\}$.

Definition 2.4. A configuration of 3-NM_1 $M = (Q, q_0, F, \Sigma, \delta)$ is an element of

$$((\Sigma \cup \{\sharp\}) \times \{+,-\})^{(3)} \times (Q \times \{+,-\}) \times N^3,$$

where N denotes the set of all nonnegative integers. The first component of a configuration $c = (\tilde{x},(q,u),(i_1,i_2,i_3))$ represents the extended input tape of M. The second component (q,u) of c represents the extended state. The third component (i_1, i_2, i_3) of c represents the input head position. If q is the state associated with configuration c, then c is said to be an *accepting configuration* if q is an accepting state. The *initial configuration* of M on input x is

$$I_M(x) = (x^-, (q_0, +), (1, 1, 1)),$$

where x^- is the special extended input tape of M such that $x^-(i_1, i_2, i_3) = (x(i_1, i_2, i_3), -)$ for each i_1, i_2, i_3 $(1 \leq i_1 \leq l_1(\tilde{x})), 1 \leq i_2 \leq l_2(\tilde{x}, 1 \leq i_3 \leq l_3(\tilde{x}))$. If M moves determinately, we call M a *three-dimensional deterministic one-marker automaton* 3-DM_1.

Definition 2.5. A *five-way three-dimensional Turing machine* is defined by the six-tuple

$$M = (Q, q_0, F, \Sigma, \Gamma, \delta),$$

where

(1) Q is a finite set of *states*;

(2) $q_0 \in Q$ is the *initial state*;

(3) $F \subseteq Q$ is the set of *accepting states*;

(4) Σ is a finite *input alphabet* ($\sharp \notin \Sigma$ is the *boundary symbol*);

(5) Γ is a finite *storage-tape alphabet* ($B \in \Gamma$ is the *blank symbol*); and

(6) $\delta \subseteq (Q \times (\Sigma \cup \{\sharp\}) \times \Gamma) \times (Q \times (\Gamma - \{B\}) \times \{$east,west, south,north,down,no move$\} \times \{$right,left,no move$\})$.

If M moves determinately (nondeterminately), we call M a *five-way three-dimensional deterministic* (*nondeterministic*) *Turing machine* FV3-DTM (FV3-NTM).

Let L: $\boldsymbol{N} \rightarrow \boldsymbol{R}$ be a function. A five-way three-dimensional Turing machine M is said to be $L(m)$ *space bounded* if for all $m \geq 1$ and for each x with $l_1(x) = l_2(x) = l_3(x) = m$, if x is accepted by M, then there is an *accepting computation path* of M on x in which M uses no more than $L(m)$ cells of the storage tape. We denote an $L(m)$ space-bounded FV3-DTM (FV3-NTM) by FV3-DTM($L(m)$) (FV3-NTM($L(m)$)).

Definition 2.6. Let $T(M)$ be the set of three-dimensional tapes accepted by a machine M, and let $\pounds[3\text{-}DM_1] = \{T | T(M)$ for some $3\text{-}DM_1$ $M\}$. $\pounds[3\text{-}NM_1]$, etc. are defined in the same way as $\pounds[3\text{-}DM_1]$.

We can easily derive the following theorem by using ordinary technique[6].

Theorem 2.1. *For any function* $L(m) \geq \log m^2$, $\pounds[FV3\text{-}NTM(L(m))] \subseteq U_{c>0}\ \pounds[FV3\text{-}DTM(2^{c(L(m))})]$.

3 Simulation of three-dimensional one-marker automata by three-dimensional Turing machines

In this section, we first investigate the sufficient spaces (i.e., upper bounds) for five-way three-dimensional Turing machines to simulate three-dimensional one-marker automata[6].

Theorem 3.1. $\pounds[3\text{-}DM_1] \subseteq \pounds[\text{FV3-NTM}(m^2 \log m^2)]$.

Proof : Suppose that a $3\text{-}DM_1$ $M = (Q, q_0, F, \Sigma, \delta)$ is given. We partition the extended states $Q \times \{+, -\}$ into disjoint subsets $Q^+ = Q \times \{+\}$ and $Q^- = Q \times \{-\}$ which correspond to the extended states when M is holding and not holding the marker in the finite control, respectively. We assume that M has a unique accepting state q_a, i.e., $|F| = 1$. In order to make our proof clear, we also assume that M begins to move with its input head on the southmost and eastmost bottom boundary symbols $\sharp$'s of input tape , i.e., position $(l+1, m+1, n+1)$ and, when M accepts an input, it enters the accepting state at the same position $(l+1, m+1, n+1)$ with the marker held in the finite control.

Suppose that an input tape x with $l_1(x) = l$, $l_2(x) = m$, and $l_3(x) = n$ is given to M. For M and x, define three types of functions $f_h^{\uparrow -}$,$f_h^{\uparrow +}$ and $f_h^{\downarrow -}$.

$f_h^{\uparrow -}(q^-, i, j) = (q'^-, i', j')$: Suppose that we make M start from the configuration $(x^-, q^-, (i, j, h-1))$, i.e., no marker existing either on the input x or in the finite control of M. After that, if M reaches the h^{th} plane of x in some time, the configuration corresponding to the first arrival is $(x^-, q'^-, (i', j', h))$,

$f_h^{\uparrow +}(q^+, i, j) = (q'^+, i', j')$: Suppose that we make M start from the configuration $(x^-, q^+, (i, j, h-1))$, i.e., holding the marker in the finite control of M. After that, if M reaches the h^{th} plane of x with its marker held in the finite control in some time (so, when M puts down the marker on the way, it must return to this position again and pick up the marker), the configuration corresponding to the first arrival is $(x^-, q'^+, (i', j', h))$,

$f_h^{\downarrow -}(q^-, i, j) = (q'^-, i', j')$: Suppose that we make M start from the configuration $(x^-, q^-, (i, j, h+1))$, i.e., no marker existing either on the input tape or in the finite control of M. After that, if M reaches the h^{th} plane of x in some time, the configuration corresponding to the first arrival is $(x^-, q^-, (i', j', h))$,

l: M never reaches the h^{th} plane of x.

Then, we can show that there exists an FV3-NTM($m^2 \log(m^2)$) M' such that $T(M') = T(M)$. Roughly speaking, while scanning from the top plane down to the bottom plane of the input , M' guesses $f_h^{\downarrow -}$, constructs $f_{h+1}^{\uparrow -}$ and $f_{h+1}^{\uparrow +}$, checks $f_{h-1}^{\downarrow -}$, and finally at the bottom plane of the input, M' decides by using $f_{t+1}^{\uparrow -}$ and $f_{t+1}^{\uparrow +}$ whether or not M accepts x. In order to record these mappings for each h, $O(m^2)$ blocks of $O(\log m^2)$ size suffice, so in total, $O(m^2 \log m^2)$ cells of the working tape suffice. It will be obvious that $T(M) = T(M')$. □

From Theorems 2.1 and 3.1, we get the following.

Corollary 3.1. $\pounds[3\text{-}DM_1] \subseteq \pounds[FV3\text{-}DTM(2^{O(m^2 \log m^2)})]$.

We next show that m^4 space is sufficient for FV3-NTM's to simulate $3\text{-}NM_1$'s. The basic idea of the proof are the same as those of Theorem 3.1.

Theorem 3.2. $\pounds[3\text{-}NM_1]\subseteq\pounds[FV3\text{-}NTM(m^4)]$.

From Theorems 2.1 and 3.1, we get the following.

Corollary 3.2. $\pounds[3\text{-}NM_1]$
$\subseteq\pounds[FV3\text{-}DTM(2^{O(m^4)})]$.

Next, we show that the algorithms described in the previous section are optimal in some sense.

Definition 3.1. Let x be in $\Sigma^{(3)}$ (Σ is a finite set of symbols) and $l_1(x)=l_2(x)=m$. For each r ($1\leq r\leq Q[l_3(x)/m^2]$) (where $Q[l_3(x)/m^2]$ denotes the quotient when $l_3(x)$ is divided by m^2),

$$x[(1,1,(r-1)m^2+1),(m,m,rm^2)]$$

is called the r^{th} (m,m)-block of x. We say that the tape x has exactly $c(m,m)$-blocks if $l_3(x)=cm^2$, where c is a positive integer.

Definition 3.2. Let (m_1,m_1),(m_2,m_2),... be a sequence of points (i.e., pairs of three natural numbers), and let $\{(m_i,m_i)\}$ denote this sequence. We call a sequence $\{(m_1,m_1)\}$ the *regular sequence of points* if $(m_i,m_i)\neq(m_j,m_j)$ for $i\neq j$.

Lemma 3.1. *Let* $\{x\in\{0,1\}^{(3)}|\ ^{\exists}m\geq 1\ [\ l_1(x)=l_2(x)=l_3(x)=m\ \&\ l_2(x)=m$ & (*each plane of* x *contains exactly one* '1') & $^{\exists}d\geq 2[(x$ *has exactly* d (m,m)-*blocks, i.e.,* $l_3(x)=dm^2$) & (*the last* (m,m)-*blocks is equal to some other* (m,m)-*block*)]]\}. *Then,*

(1) $T_1\in\pounds[3\text{-}DM_1]$, *but*

(2) $T_1\notin\pounds[FV3\text{-}DTM(2^{L(m)})]$ (*so,* $T_1\notin\pounds[FV3\text{-}NTM(L(m))]$) *for any function* $L(m)$ *such that*

$$lim_{i\to\infty}[L(m_i)/(m_i^2\log(m_i^2))]=0$$

for some regular sequence of points $\{(m_i,m_i)\}$.

Proof : (1) We construct a 3-DM_1 M accepting T_1 as follows. Given an input x with $l_1(x)=l_2(x)=l_3(x)=m$, M first checks, by sweeping plane by plane, that each plane of x contains exactly one '1,' and M then checks, by making a zigzag of 45°-direction from top plane to bottom plane, that x has exactly d (m,m)-blocks for some integer $d\geq 2$. After that, M tests by utilizing its own marker whether the last (m,m)-block is identical to some other (m,m)-block: In order to check whether the p^{th} plane of the h^{th} (m)-block is identical to the p^{th} plane of the last (m)-block ($1\leq p\leq m^2$, $1\leq h\leq d$), M first puts the marker on the position $(i,j,m^2(h-1)+p)$. After that, M vertically moves down until encounters the bottom boundary, after which it moves up (m^2-p) plane by making a zigzag of 45°-direction. At this time, M arrives at the p^{th} plane of the last (m,m)-block. M then finds the '1' position on the plane and moves up vertically from this position. In this course, each time M meets a '1' position, it checks whether or not there is a marker on the plane (containing the '1' position). In this way, M enters an accepting state just when it finds out some (m,m)-block, each of whose planes is identical to the corresponding plane of the last (l,m)-block. It will be obvious that $T(M)$=T_1.

(2) Suppose to the contrary that there exists an FV3-DTM($2^{L(m)}$) M accepting T_1, where $L(m)$ is a function such that

$$lim_{i\to\infty}[L(m_i)/(m_i^2\log(m_i^2))]=0$$

for some regular sequence of points $\{(m_i,m_i)\}$. Then, by using the well-known technique [6], we can get the desired result. □

From Lemma 3.1., we can conclude as follows.

Theorem 3.3. *To simulate* $3-DM_1$*'s, (1) FV3-NTM's require* $\Omega(m^2\log(m^2))$ *space and (2) FV3-DTM's require* $2^{\Omega(m^2\log(m^2))}$ *space.*

Next, we can get the following lemma by using the same technique as in the proof of Lemma 3.1.

Lemma 3.2. *Let* T_2=$\{x\in\{0,1\}^{(3)}|\ ^{\exists}m\geq 1\ [\ l_1(x)=l_2(x)=l_3(x)=m\ \&\ \&\ ^{\exists}d\geq 2\ [(x$ *has exactly* d (m,m)-*blocks, i.e.,* $l_4(x)=dm^2$) & (*the last* (m,m)-*block is different from any other* (m,m)-*block*)]]\}. *Then,*

(1) $T_2\in\pounds[3\text{-}NM_1]$, *but*

(2) $T_2\notin\pounds[FV3\text{-}DTM(2^{L(m)})]$ (*so,* $T_2\notin\pounds[FV3\text{-}NTM(L(m))]$) *for any function L such that* $lim_{i\to\infty}[L(m_i)/(m_i^4)]=0$ *for some regular sequence of points* $\{(m_i,m_i)\}$.

From Lemma 3.2., we can conclude as follows.

Theorem 3.4. *To simulate* 3-NM_1*'s,*

(1) *FV3-NTM's require* $\Omega(m^4)$ *space, and*
(2) *FV3-DTM's require* $2^{\Omega(m^4)}$ *space.*

4 Recognizability of interlocking components in three-dimensional images

In this paper, we show that interlocking components are not recognized by any space-bounded three-dimensional Turing machines.

First of all, we consider a three-dimensional input tape T_3 that is 7 units in thickness. So, for some m, T_3=\{ (i_1,i_2,i_3) | $1\leq i_1,i_2\leq$ m+2,$1\leq i_3\leq 7$ \}.
Fig.3(a)represents T_3. Now we define two different 5 $\times$ 5 $\times$ 5 patterns as shown in Fig.3(b)(c). Then we consider an arbitrary n-by-n matrix of those 5 $\times$ 5 $\times$ 5 patterns (see Fig.3).

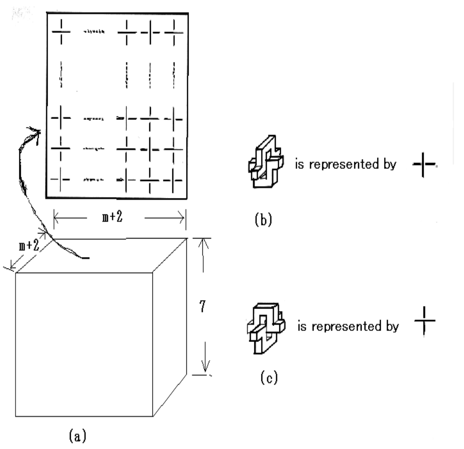

Fig. 3: Three-dimensional input tape including interlocking components T_3[2].

Then, we can get the following lemma from Lemma 2.1 in [2].

Lemma 4.1. 3-*DM*, *cannnot recognize interlocking components of an arbitrary given digital picture.*

From Theorem 3.3 and Lemma 4.1, we can get the following.

Theorem 4.1. *Interlocking components are not accepted by any* $FV3$-DTM $(L(m))(FV3$-$NTM(L(m)))$ *for any function* $L(m)$ *such that* $\lim_{m\to\infty}$[L(m)/2m$^2 log\, m^2$]=0 $(\lim_{m\to\infty}[L(m)/m^2 log\ m^2] = 0)$.

Next, we can get the following lemma by using a technique similar to that in the proof of Lemma 2.1 in [2].

Lemma 4.2. 3-ND_1 *cannnot recognize interlocking components of an arbitrary given digital picture.*

From Theorem 3.4 and Lemma 4.2, we can get the following.

Theorem 4.2. *Interlocking components are not accepted by any* $FV3$-$DTM(L(m))$ $(FV3$-$NTM(L(m)))$ *for any function* $L(m)$ *such that* $\lim_{m\to\infty}[L(m)/2^{m^4}] = 0 \quad (\lim_{m\to\infty}[L(m)/m^4] = 0)$.

5 Conclusion

In this paper, we investigated recognizability of topological components by three-dimensional automata, and showed that interlocking components are not recognized by any space-bounded three-dimensional deterministic or nondeterministic Turing machines. By the way, what is the situation for a two or three marker automata, or for alternation (see [1])? This question seems very intersting. We will deal with the problem in further papers.

References

[1] A.K. Chandra , D.C. Kozen, and L.J. Stockmeyer, Alternation, *J. of ACM*, 28(1):114-133(1981).

[2] A.Nakamura and K.Aizawa, Detection of ic interlocking components in three-dimensional digital pictures, Imformation Sciences 40:143-153(1986).

[3] A.Rosenfeld and K.C. Kak, Digital Picture Processing, Academic Press, New York(1976).

[4] A.Rosenfeld, Picture Languages, Academic Press, New York(1979).

[5] A.Rosenfeld, Three-dimensionl digital topology, Information and Control 50:119-127(1981).

[6] M.Sakamoto, Three-dimensional alternating Turing machines, *Ph. D. Thesis*, *Yamaguchi University* (1999).