Three-Dimensional Parallel Turing Machines

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Abstract

Informally, a parallel Turing machine (PTM) proposed by Wiedermann is a set of identical usual sequential Turing machines (STM's) cooperating on two common tapes – storage tape and input tape. Moreover, STM's which represent the individual processors of the parallel computer can multiply themselves in the course of computation. On the other hand, during the past about twenty-five years, automata on a three-dimensional tape have been proposed as computational models of three-dimensional pattern processing and several properties of such automata have been obtained. In this paper, we propose a three-dimensional parallel Turing machine (3-PTM), and investigate its some properties. Especially, we deal with a hardware-bounded 3-PTM, whose inputs are restricted to cubic ones. We believe that this machine is useful in measuring the parallel computational complexity of three-dimensional images.

Key Words: computational complexity, hardware-bounded computation, nondeterminism, parallel Turing machine, three-dimensional automaton

1 Introduction

A parallel Turing machine (PTM) is a set of identical sequential Turing machines (STM's) cooperating on two common tapes – storage tape and input tape [5]. Moreover, STM's which represent the individual processors of the parallel computer can multiply themselves in the course of computation. In [5] it is shown, for example, that every PTM can be simulated by an STM in polynomial time, and that the PTM cannot be simulated by any sequential Turing machine in linear space.

In [1], two-dimensional version of PTM was investigated. On the other hand, due to the advances in many application areas such as computer vision, robotics, and so forth, it has become increasingly apperant that the study of three-dimensional pattern processing has been of crucial importance. Thus, we think that the research of three-dimensional automata as a computational model of three-dimensional pattern processing has also been meaningful. During the past about twenty-five years, automata on a three-dimensional tape have been proposed and several properties of such automata have been obtained. In this paper, we propose a three-dimensional parallel Turing machine (3-PTM), and investigate its some properties. Especially, we deal with a hardware-bounded 3-PTM, a variant of the 3-PTM, whose inputs are restricted to cubic ones. The hardware-bounded 3-PTM is a 3-PTM, the number of whose processors is bounded by a constant or variable depending on the size of inputs. The investigation of hardware-bounded 3-PTM's is more useful than that of 3-PTM's from the practical point of view.

2 Preliminaries

Definition 2.1. Let Σ be a finite set of symbols, a *three-dimensional tape* over Σ is a three-dimensional rectangular array of elements of Σ. The set of all three-dimensional tapes over Σ is denoted by $\Sigma^{(3)}$. Given a tape $x \in \Sigma^{(3)}$, for each integer j $(1 \leq j \leq 3)$, we let $l_j(x)$ be the length of x along the jth axis. The set of all $x \in \Sigma^{(3)}$ with $l_1(x) = n_1$,$l_2(x) = n_2$ and $l_3(x)$=n_3 is denoted by $\Sigma^{(n_1,n_2,n_3)}$. When $1 \leq i_j \leq l_j(x)$ for each j $(1 \leq j \leq 3)$, let $x(i_1, i_2, i_3)$ denote the symbol in x with coordinates (i_1, i_2, i_3). Furthermore, we define

$$x[(i_1, i_2, i_3),(i'_1, i'_2, i'_3)],$$

only when $1 \leq i_j \leq i'_j \leq l_j(x)$ for each integer j $(1 \leq j \leq 3)$, as the three-dimensional input tape y satisfying the following conditions:

(1) for each j $(1 \leq j \leq 3)$, $l_j(y) = i'_j - i_j + 1$;

(2) for each r_1, r_2, r_3 ($1 \leq r_1 \leq l_1(y)$, $1 \leq r_2 \leq l_2(y)$, $1 \leq r_3 \leq l_3(y)$), y (r_1,r_2,r_3) = x (r_1+i_1-1, r_2+i_2-1, r_3+i_3-1). (We call $x[(i_1,i_2,i_3),(i'_1,i'_2,i'_3)]$ the $[(i_1,i_2,i_3),(i'_1,i'_2,i'_3)]$-*segment* of x.)

For each $x \in \Sigma^{(n_1,n_2,n_3)}$ and for each $1 \leq i_1 \leq n_1$, $1 \leq i_2 \leq n_2$, $1 \leq i_3 \leq n_3$, x[(i,1,1),(i_1,n_2,n_3)], x[(1,i_2,1),(n_1,i_2,n_3)], x[(1,1,i_3),(n_1,n_2,i_3)], x[(i,1,i_3),(i_1,n_2,i_3)], and x[(1,i_2,i_3),(n_1,i_2,i_3)] are called the i_1th (2-3) *plane of* x, the i_2th (1-3) *plane of* x, the i_3th (1-2) *plane of* x, the i_1th *row on the* i_3th (1-2) *plane of* x, and the i_2th *column on the* i_3th (1-2) *plane of* x, and are denoted by x(2-3)i_1, x(1-3)i_2, x(1-2)i_3, $x[i_1,*,i_3]$, and $x[*,i_2,i_3]$, respectively.

Definition 2.2. *Three-dimensional parallel Turing machine* (denoted by 4-PTM) is a 10-tuple M = (Q, E, U, q_s, q_0, Σ, Γ, F, δ_n, δ_f), where

(1) $Q = E \cup U \cup \{q_0\}$ is a finite set of *states*;

(2) E is a finite set of *nondeterministic states*;

(3) U is a finite set of *fork states*;

(4) q_s is the *quiescent state*;

(5) $q_0 \in Q$ - $\{q_s\}$ is the *initial state*;

(6) Σ is a finite *input alphabet* ($\# \notin \Sigma$ is the *boundary symbol*);

(7) Γ is a finite *storage tape alphabet* containing the special *blank* symbol B;

(8) $F \subseteq Q - \{q_s\}$ is the set of *accepting states*;

(9) $\delta_n : E \times (\Sigma \cup \{\#\}) \times \Gamma \rightarrow 2^{(Q-\{q_s\})\times(\Gamma-\{B\})\times D_{in}\times D_s}$ (where D_{in} = {east, west, south, north, up, down, no more} and D_s = {left, right, no more}) is a *next nondeterministic move function*; and

(10) $\delta_f : U\times (\Sigma\cup\{\#\}) \times \Gamma \rightarrow U_{1\leq k\leq\infty} ((Q-\{q_s\})\times(\Gamma - \{B\}) \times D_{in} \times D_s)$ is a *next fork more function* with the restriction that for each $q \in U$, each $a \in \Sigma \cup \{\#\}$, and each $A \in \Gamma$, if $\delta(q,a,A) = ((p_1,c_1,d_{11},d_{21}), (p_2,c_2,d_{12},d_{22}), \ldots, (p_k,c_k,d_{1k},d_{2k}))$, then $c_1 = c_2 = \ldots = c_k$.

As shown in Figure.1, M has a read-only three-dimensional rectangular input tape with boundary symbols "#'s", and one semi-infinite storage tape (extended to the right), initially filled with the blank symbols. Furthermore, M has inifinite processors, $P_1, P_2, \ldots$, each of which has its input head and storage-tape head. M starts in the situation that (1) the processors P_1 is in the initial state q_0 with its input head on the upper northwestmost corner of the input tape and with its storage-tape head on the leftmost cell of the storage tape, and (2) each of other processors is in the quiescent state q_s with its input head on the upper northwestmost corner of the input tape and with its storage-tape head on the leftmost cell of the stroage tape.

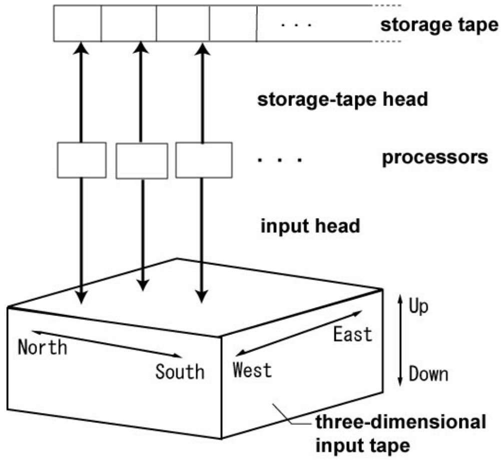

Figure 1: Three-dimensional parallel Turing machine.

Five-way three-dimensional parallel Turing machine (denoted by FV3-PTM) is a 3-PTM, input heads of whose processors cannot move up. In this paper, we are concerned with three-dimensional parallel Turing machines whose input tapes are restricted to cubic ones. Let $L : \mathbf{N} \rightarrow \mathbf{N}$ and $H : \mathbf{N} \rightarrow \mathbf{N}$ be functions. A 3-PTM (FV3-PTM) M is called $L(n)$ *space-bounded* if for any $n \geq 1$ and for any input tape x with $l_1(x) = l_2(x) = l_3(x) = n$, M on x uses at most $L(n)$ cells of the storage tape, and M is $H(n)$ *hardware-bounded* if for any $n \geq 1$ and for any input tape x with $l_1(x) = l_2(x) = l_3(x) = n$, M on x activates at most $H(n)$ processors. We use the follwing notations:

· D3-PTM($L(n), H(n)$): the class of sets of cubic tapes accepted by $L(n)$ space-bounded and $H(n)$ hardware-bounded deterministic 3-PTM's · N3-PTM($L(n), H(n)$): the class of sets of cubic tapes accepted by $L(n)$ space-bounded and $H(n)$ hardware-bounded nondeterministic 3-PTM's · DFV3-PTM($L(n), H(n)$): the class of sets of cubic tapes accepted by $L(n)$ space-bounded and $H(n)$ hardware-bounded deterministic FV3-PTM's · NFV3-PTM($L(n), H(n)$): the class of sets of cubic tapes accepted by $L(n)$ space-bounded and $H(n)$ hardware-bounded nondeterministic FV3-PTM's

3 Main Results

This section mainly investigates accepting powers of FV3-PTM's, based on hardware complexity.

A function $L : \mathbf{N} \to \mathbf{N}$ is *fully space constructible* by a *k head one-dimensional deterministic Turing machine* if there is a k head one-dimensional deterministic Turing machine [4] M such that for any $n \geq 1$ and any input word x of length n, M on x marks off exactly $L(n)$ cells of the storage tape and halts. (In this case, we say that M *constructs* the function L.)

Theorem 3.1. Let $H : \mathbf{N} \to \mathbf{N}$ *be a function which satisfies the following* (1), (2), and (3), where $k \geq 1$ *is an integer*:

(1) H *is fully space constructible by a k head one-dimensional deterministic Truing machine*;

(2) $\exists_{n_0} \in \mathbf{N}$, $\forall_n \geq n_0$ $[H(n) \geq k]$;

(3) $\binom{H(n)}{2} \leq \frac{n}{2}(n \leq 2)$.

Furthermore, let H' : $\mathbf{N} \to \mathbf{N}$ *and* $L : \mathbf{N} \to \mathbf{N}$ be functions which satisfy the following (4) and (5):

(4) $\exists_{n_0} \in \mathbf{N}$, $\forall_n \geq n_0$ $[\binom{H'(n)}{2} \leq \binom{H(n)}{2}]$;

(5) max $\{ H'(n)^2\binom{H(n)}{2}log\, n,$
$H'(n)^2\binom{H(n)}{2}log\, L(n),$
$L(n)H'(n)\binom{H(n)}{2} \} = o(n)$.

Then,

*DFV*3*-PTM*$(H(n)$,$H(n))$ $-$ *NFV*3*-PTM*$(L(n)$, $H'(n)) \neq \phi$.

Proof: Let $T(H)$ be the following set depending on the function H in the theorem:

$T(H) = \{ x \in \{0,1\}^{(3)} \mid \exists_n \geq 2 \binom{H(n)}{2}$ $[l_1(x)$=$l_2(x)$ =$l_3(x)$=$l_4(x)$=n & $\forall_i(1 \leq i \leq \binom{H(n)}{2})$) [the ith plane of x is identical with the $(2\binom{H(n)}{2}$+1-i)th plane of x]] }.

To prove the theorem, we show that $T(H) \in DFV3$-PTM $(H(n), H(n))$ $-$ $NFV3$-PTM $(L(n), H(n))$. $T(H)$ is accepted by an $H(n)$ space-bounded and $H(n)$ hardware-bounded *DFV*3*-PTM* M which acts as follows. Suppose that an input tape x with $l_1(x) = l_2(x) = l_3(x) = n$ is presented to M. Let M_1 be a k head one-dimensional deterministic Turing machine which constructs the function H. By simulating the action of M_1 on the first plane of x, the first k processors $P_1, P_2, \ldots, P_k$ of M mark off exactly $H(n)$ cells of the storage tape.

After this, each processor $P_i(2 \leq i \leq k)$ positions its storage-tape head on the ith cell (from the left) of the storage tape, and processor P activates processors $P_{k+1}, P_{k+2}, \ldots, P_{H(n)}$ in such a way that for each j $(k+1 \leq j \leq H(n))$, the storage-tape head of P_j is positioned on the jth cell (from the left) of the storage tape. Then P_1 positions the input head at the northwsetmost corner of the $(2\binom{H(n)}{2} + 2 - H(n))$th plane of x, which for each i $(2 \leq i \leq H(n))$, P_i positions the input head on the northwestmost corner of the $(H(n) - i + 1)$th plane of x. And P_1 systematically traverses the $(2\binom{H(n)}{2} + 2 - H(n))$th plane, $\ldots$, the $2\binom{H(n)}{2}$th plane (from the first column to the last column in each plane, and from the first row to the last row in each column), and compares these planes with the $(H(n)-1)$th plane, $\ldots$, the first plane, respectively, by using the information from $P_2, P_3, \ldots, P_{H(n)}$.

These input heads are then positioned at the northwestmost end of the $H(n)$th plane of x. The same procedure is used inductively to verify that $H(n)$th plane through the $(2\binom{H(n)}{2} + 1 - H(n))$th plane has a desired form.

Next, we show that $T(H) \notin NFV3$-PTM $(L(n)$, $H'(n))$. Suppose to the contrary that there is an $NFV3$-PTM $(L(n)$,$H'(n))$ M' accepting $T(H)$. Let s and t be the numbers of states of the finite control of each processor and storage tape symbols of M', respectively. For large $n \geq 2\binom{H(n)}{2}$, let

$V(n) = \{ x \in \{0,1\}^{(3)} \mid l_1(x) = l_2(x) = l_3(x) = n$ & $\forall_i$ $(1 \leq i \leq \binom{H(n)}{2})$) [the ith plane of x is identical with the $(2\binom{H(n)}{2} + 1 - i)$th plane of x] & $[(1, 1, 2\binom{H(n)}{2} + 1)$, $(n, n, n)] \in \{0\}^{(3)}$ }.

Below, we consider the computation of M' on input tapes in $V(n)$. Clearly, each tape x in $V(n)$ is in $T(H)$, and so x is accepted by M'.

A configuration of M' is an infinite-tuple (α, $((i_1, j_1, k_1), q_1, h_1)$, $((i_2, j_2, k_2), q_2, h_2)$, ..., $((i_m, j_m, k_m)$, q_m, $h_m)$, $\ldots$) where α is the non-blank contents of the storage tape of M', and for each $m \geq 1$, (i_m, j_m, k_m), q_m and h_m are the input head position, the state of the finite control and the position of storage-tape head of the mth processor of M', respectively. The type of a configuration C =(α, $((i_1, j_1, k_1), q_1, h_1)$, $((i_2, j_2, k_2), q_2, h_2)$, ..., $((i_m, j_m, k_m)$, q_m, $h_m)$, $\ldots$), denoted by Type(C), is an infinite-tuple $([i_1], ..., [i_m], \ldots)$, where for each $m \geq 1$,

$$[i_m] = \begin{cases} i_m & \text{if } i_m \leq \binom{H(n)}{2} \\ 2\binom{H(n)}{2} & \textit{otherwise.} \end{cases}$$

Let $c_1(x)$, $c_2(x)$, ..., $c_{l_x}(x)$ be the sequence of configurations of M' during an (arbitrary selected) accepting computation of M' on a tape x in $V(n)$. Here l_x is the length of this computation. Let $d_1(x), d_2(x), \ldots, d_{l'x}(x)$ be the subsequence obtained by selecting $c_1(x)$ and all subsequent $c_i(x)$'s such that $Type(c_i(x)) \neq Type(c_{i+1}(x))$. We call $d_1(x), d_2(x), \ldots, d_{l'x}(x)$ the pattern of x. Let $p(n)$ be the number of possible pattern of M' on x in $V(n)$. Since $L'_x \leq H'(n)(2\binom{H(n)}{2} - 1) + 1 \equiv Q(x)$ (note that M' uses at most $H'(n)$ processors when it reads tapes in $V(n)$), we get the following inequality:

$$p(x) \leq ((s(n+1)(n+1)(n+1)L(n))^{H'(m)t^{L(n)}})^{Q(n)}.$$

Now we classify the tapes in $V(n)$ according to their patterns. There must exist a pattern $\hat{d_1}, \hat{d_2}, \ldots, \hat{d_l}$ which corresponds to a set $S(n)$ of at least $2^{n\times n\times\binom{H(n)}{2}}$ / $p(n)$ tapes in $V(n)$. Since $\binom{H'(n)}{2} \leq \binom{H(n)}{2}$ (from condition (4) in the theorem), the same observation as in the proof of Theorem 3 in [2] reveals that for any computation of M' on an $x \in V(n)$, there exists an index i such that the ith plane of x and the $(2\binom{H(n)}{2}+1-i)$th plane of x are never being read simultaneously.

Let i_0 be such a value for the pattern $\hat{d_1}, \hat{d_2}, \ldots, \hat{d_l}$. we now define a binary relation E on tapes in $S(n)$ as follows: For each u and v in $S(n)$, let

${}_uE_v \Leftrightarrow \forall_i(\notin \{i_0, i_0, 2\binom{H(n)}{2}+1-i_0)$ [ith planes of u and v are identical].

Obviously the relation E is an equivalence relation, and there are $q(n) = 2^{n^2(\binom{H(n)}{2}-1)}$ E-equivalence classes of tapes in $S(n)$. From condition (5) in the theorem, we can easily show that $|S(n)| > q(n)$ for large n. Therefore, there exist two different tapes in $S(n)$ which belong to the same equivalence class. Let x and y be such two different tapes in $S(n)$. And let z be the tape obtained from x by replacing the $(2\binom{H(n)}{2}+1-i_0)$th plane with the $(2\binom{H(n)}{2}+1-i_0)$th plane of y. By an argument similar to that in the proof of theorem 1 in [6], it can be shown that there is an accepting computation of M' on z. Consequently, z must be accepted by M'. This contradicts the fact z is not in $T(H)$. □

We consider the following functions:

$\cdot\ log^{(1)}n = \begin{cases} 0 & (n=0) \\ \lceil logn \rceil & (n \geq 1), \end{cases}$

and for each $r \geq 1$,

$\cdot\ log^{(r+1)}n = log^{(1)}(log^{(r)}n)$.

It is shown in [3] that the function $log^{(k)}n$ $(k \geq 1)$ are fully space-constructible by three head one-dimensional deterministic Turing machines. From this fact and Theorem 3.1, we have:

Corollary 3.1. *For each* $k \geq 3$,
$DFV3\text{-}PTM$ $(log^{(k)}n, log^{(k)}n)$ $-$ $NFV3\text{-}PTM$ $(log^{(k)}n,\ log^{(k+1)}n) \neq \phi$.

Corollary 3.2. *Fpr each* $X \in \{D, N\}$ and each $k \geq 3$,
$XFV3\text{-}PTM$ $(log^{(k)}n, log^{(k+1)}n) \subseteq XFV3\text{-}PTM$ $(log^{(k)}n, log^{(k)}n)$.

Letting $H(n) = k+1$ (where k is a positive integer), H'$(n) = k$, and $L(n) = o(n)$ in Theorem 3.1, we have :

$DFV3\text{-}PTM$ $(k+1, k+1) - NFV3\text{-}PTM$ $(0(n), k) \neq \phi$.

From this and from the obvious fact that

$DFV3\text{-}PTM$ $(k+1, k+1) = DFV3\text{-}PTM(1, k+1)$, we have the following corollary.

Corollary 3.3. *For any integer* $k \geq 1$,
$DFV3\text{-}PTM$ $(1, k+1) - NFV3\text{-}PTM$ $(o(n), k) \neq \phi$.

4 Conclusion

This paper investigated fundamental properties of three-dimensional parallel Turing machines with bounded number of processors. We conclude the paper by giving several open problems.

(1) What is a relationship between the accepting powers of $FV3\text{-}PTM$'s and $3\text{-}PTM$'s?

(2)What is a relationship between the accepting powers of deterministic and nondeterministic $FV3\text{-}PTM$'s?

(3) What is a hierarchy of the accepting powers of $FV3\text{-}PTM$'s, based on the hardware complexity depending on the side-length of input tapes?

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