# On the periodic sequence of a discrete sliding mode control system for a single-link robot arm

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**Abstract:** A novel sufficient condition for a sampled-data system with constant-gain discrete sliding mode controller (SMC) to be globally uniformly ultimately bounded (GUUB) is proposed. It is shown that the stability of the overall system can be known by checking out the one parameter. Simulation results for a single-link robot arm are presented to verify the feasibility of the proposed method.

**Keywords:** Sampled-Data System, Discrete Sliding Mode Control, GUUB.

## 1. INTRODUCTION

Almost all of studies of sliding mode control (SMC) have been proposed in the continuous-time domain[1-2]. In the actual system, however, controller is implemented in the discrete time domain since they use micro-processors and/or digital computers. Recently, discrete-time sliding mode control (DSMC) has been studied extensively to address various controllers using specific principles [3-6]. However, the research of discretizing a continuous-time SMC for digital implementation has not been fully explored. Furthermore, it is also well known that a control system designed in the continuous-time domain may become unstable after sampling.

Recently chaotic behaviors were found in discretizing continuous SMC systems by X. Yu [7-8]. Yu and G. Chen proposed the sufficient conditions for discretized system to be GUUB [9]. But these sufficient conditions can be only applied to limited sampling period and specialized cases.

In this paper, therefore, a novel sufficient condition for discrete sliding mode controller (SMC) to be globally uniformly ultimately bounded (GUUB) is proposed. It is shown that the stability of the overall system can be known by checking out the magnitude of one parameter which is an element of the discretized system matrix. The ultimate bounds of the system state variables are also derived. Simulation results for a single-link robot arm are presented to show the effectiveness of the proposed method.

## 2. SYSTEM DESCRIPTION

Consider a second-order system of the following form

$$\dot{x} = Ax + bu = \begin{bmatrix} 0 & 1 \\ -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \,, \qquad (1)$$

where $x \in R^2$ is the state vector, $u \in R^1$ is the system input, and $a_1, a_2$ are elements of a system matrix. Let the sliding surface be $\sigma = c^T x = [c_1 \; 1] x$, where $c_1 > 0$ is assumed to be designed such that the sliding dynamics, $\sigma = 0$, are asymptotically stable. From $\dot{\sigma} = 0$, we can easily obtain the equivalent control law as

$$\dot{\sigma} = c^T \dot{x} = c^T Ax + c^T bu = 0 \;\Rightarrow\; u_{eq} = -(c^T b)^{-1} c^T Ax.$$

From the sliding mode existence condition, $\sigma\dot{\sigma} = 0$, we have the following equivalent control based SMC:

$$\begin{aligned} u &= u_{eq} + u_s \\ &= -(c^T b)^{-1} c^T Ax - \alpha (c^T b)^{-1} \operatorname{sgn}(\sigma(x)), \end{aligned} \qquad (2)$$

where $\alpha > 0$ is a control gain, and $\operatorname{sgn}(\cdot)$ is a signum function. It's assumed that $c^T b$ is nonsingular.

To discretize the overall system, we convert the continuous-time system (1) under the zero-order hold (ZOH) to the discrete-time system

$$x(k+1) = e^{Ah} x(k) + \int_0^h e^{A\tau} d\tau b u(k), \qquad (3)$$

where

$$\begin{aligned} u(k) &= u_{eq}(k) + u_s(k) \\ &= -c^T Ax(k) - \alpha \cdot \operatorname{sgn}(\sigma(x(k))), \end{aligned} \qquad (4)$$

$h$ is a sampling period, and the index $k$ indicates the $k$- th sample.

As the system state $x(k)$ evolves, the switching function $\operatorname{sgn}(\sigma(x(k)))$ forms a sequence of binary values of $-1$ and $+1$ . For simplicity, we denote $\operatorname{sgn}(\sigma(x(k)))$ as $s_k \in \{-1, 1\}$ .

Then the discretized system can be described by

$$\begin{aligned} x(k+1) &= \Phi x(k) + \alpha \Gamma s_k \\ &= \begin{bmatrix} 1 & v(h) \\ 0 & d(h) \end{bmatrix} x(k) + \alpha \begin{bmatrix} \gamma_1(h) \\ \gamma_2(h) \end{bmatrix} s_k, \end{aligned} \qquad (5)$$

where $\Phi = e^{Ah} - \int_0^h e^{A\tau} d\tau b c^T A$, $\Gamma = \int_0^h e^{A\tau} d\tau b$.

Equation (5) can be rearranged as

$$x_1(k+1) = x_1(k) + v x_2(k) - \gamma_1 \alpha s_k \,, \tag{6}$$

$$x_2(k+1) = d x_2(k) - \gamma_2 \alpha s_k \,. \tag{7}$$

## 3. STABILITY CONDITION OF DISCRETE SLIDING MODE CONTROLLER

Generally, the asymptotic stability can be guaranteed if the sliding mode controller with a constant gain is implemented in the continuous-time domain. For the discrete-time system, however, the ultimate boundedness can be ensured. In the following theorem, we derive conditions for the stability of the closed-loop system with discrete SMC (4).

**Theorem 1:** For the discretized systems (6)~(7) with the discrete SMC (4), the overall system is globally uniformly ultimately bounded (GUUB) if

$$|d(h)| < 1 \,. \tag{8}$$

and

$$\frac{v(h)\gamma_2(h)}{1-d(h)} + \gamma_1(h) > 0. \tag{9}$$

Furthermore, ultimately bound of the system state variables are given by

$$\lim_{k\to\infty} |x_1(k)| \le |\gamma_1 \alpha| + \frac{(c_1^{-1} - v)|\gamma_2 \alpha|}{1-|d|}, \tag{10}$$

$$\lim_{k\to\infty} |x_2(k)| \le \frac{|\gamma_2 \alpha|}{1-|d|}. \tag{11}$$

**Proof:** From (7), It is clear that (8) has to be satisfied because the pole of the system (7) should be located inside the unit circle. It is obvious that the ultimate bound of $x_2$ can be obtained as (11). When the state $x_2$ reaches its ultimate bound, we say $x_2$ is on the equilibrium line, (6) can be rewritten as

$$x_1(k+1) = x_1(k) - \frac{v\gamma_2 \alpha s_k}{1-d} - \gamma_1 \alpha s_k \,. \tag{12}$$

In order to the state $x_1$ converges to the sliding surface, the last two terms of (12) should satisfy the following inequality:

$$-\frac{v\gamma_2 \alpha s_k}{1-d} - \gamma_1 \alpha s_k \Rightarrow \frac{v\gamma_2}{1-d} + \gamma_1 > 0. \tag{13}$$

The ultimate bound of the state $x_1$ can be derived by considering the switching points – intersection of the sliding surface and the equilibrium lines. Since the points are on the sliding surface and equilibrium line, $x_2$ should have a value of its ultimate bound, and $x_1$ has to satisfy

$$x_1(k) = -c_1^{-1} x_2(k). \tag{14}$$

Substituting (14) into (6) gives

$$x_1(k+1) = \frac{(c_1^{-1} - v)|\gamma_2 \alpha| s_k}{1-|d|} - \gamma_1 \alpha s_k \,. \tag{15}$$

From (15), therefore, the ultimate bound of $x_1$ can be obtained as (10).

■

Theorem 1 needs two conditions, (8) and (9), to check the stability of the discretized system. Actually, the condition (9) contains four variables, $v$, $d$, $\gamma_1$, $\gamma_2$, whereas the condition (8) is composed of only one variable $d$. In the following theorem, we show that it is sufficient to check the condition (8) to guarantee the stability of the overall system.

**Theorem 2:** For the closed-loop system (5), if the inequality (8) holds, then the inequality (9) is also satisfied. That is, $|d| < 1$ implies $\frac{v\gamma_2}{1-d} + \gamma_1 > 0$.

**Proof:** Due to the page limitation, the proof is omitted.

■

**Remark 1:** From Theorem 1 and 2, the stability of the overall system can be obtained by checking up the magnitude of $d$.

**Remark 2:** In the conventional digital system, the stability of the closed-loop system depends on the sampling period, and is given by one inequality. For the constant-gain sliding mode controller for the sampled-data system, however, the stable region of the sampling period may be composed of several supports. For example, for a single-link robot arm, the stable region for the sampling period $h$ is given as a shaded region as in Figure 1.

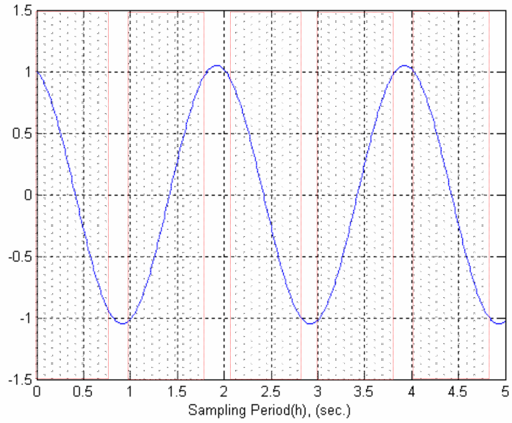


**Fig. 1** Stable sampling period. ($|d(h)| < 1$).

## 4. SIMULATION STUDIES

Consider the model of a pendulum without damping. A second-order system of the pendulum can be represented as

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{1}{m}u. \tag{55}$$

where g is the acceleration of gravity, $l = 1\,m$ and $m = 1\,kg$ are the length and mass of pendulum. Assume that $\theta$ is small enough that $\sin\theta \approx \theta$.

Figures 2~5 are results when $h = 0.1$ second. From Figure 1, it's clear that the system is stable for the sampling period. Figure 2 shows the phase portrait. It's very similar to that of continuous-time system. In addition, it is shown that, in the steady state, the system states are bounded by the region given in Theorem 1:

$|x_1(\infty)| \le 0.61,\ |x_2(\infty)| \le 0.67$ .

It is also can be seen in Figures 3~4.

Figures 6~9 are results when $h = 1.5$ second. From Figure 1, it's obvious that the system is stable for the sampling period. For this sampling period, in the steady state, the profile of the state variables show a kind of limit cycle; periodic solution (Fig. 6), and their bounds given in Theorem 1 are as follows:

$|x_1(\infty)| \le 0.86,\ |x_2(\infty)| \le 0.46$

Figures 7~8 show that the state variables show the period-12 profiles.

Figures 10~12 are results when $h = 0.9$ second. From Figure 1, it's easy to know that the system is unstable for the sampling period although the sampling period is shorter than the another stable one, $h = 1.5$. The unstable profile can be seen in Figures 10~11.

## 5. CONCLUSION

In this paper, a condition to be GUUB for the sampled-data system with discrete SMC has been presented. The proposed scheme gives a simple way to check up the stability of the closed-loop system for a given sampling period.

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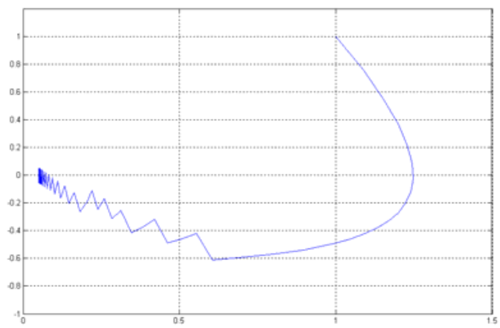

**Fig. 2** Phase portrait when $h = 0.1$

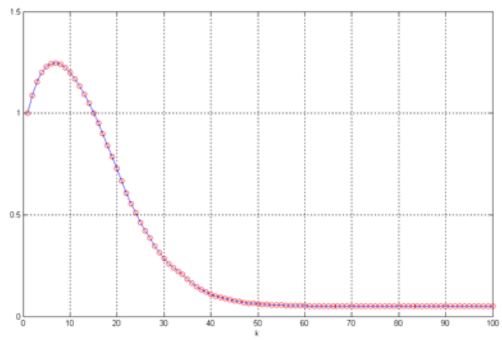

**Fig. 3** State variable $x_1$ when $h = 0.1$

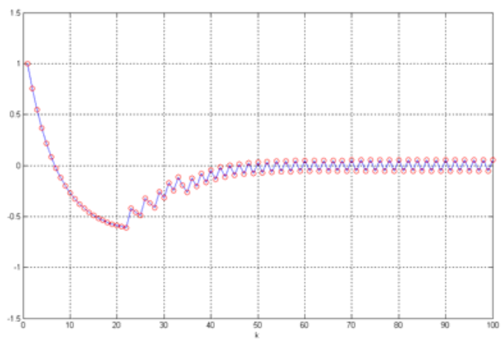

**Fig. 4** State variable $x_2$ when $h = 0.1$

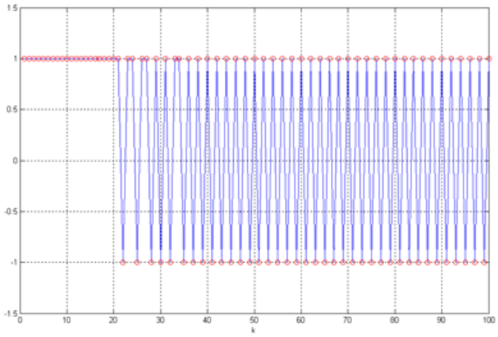

**Fig. 5** Switching variable $s$ when $h=0.1$

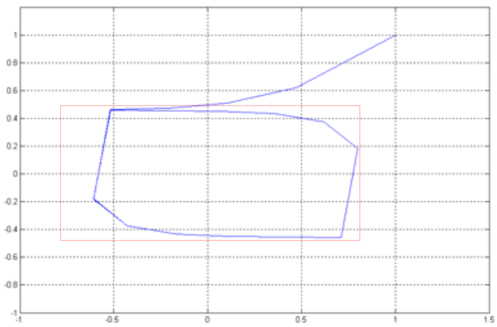

**Fig. 6** Phase portrait when $h=1.5$

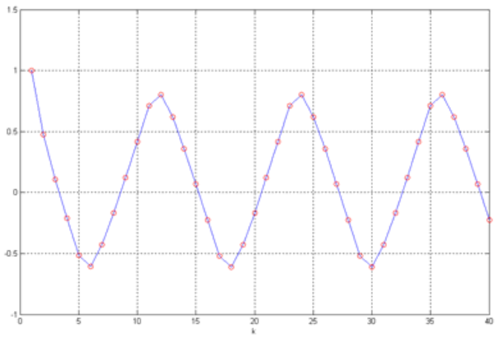

**Fig. 7** State variable $x_1$ when $h=1.5$

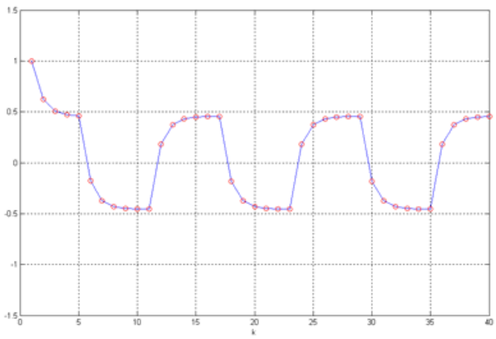

**Fig. 8** State variable $x_2$ when $h=1.5$

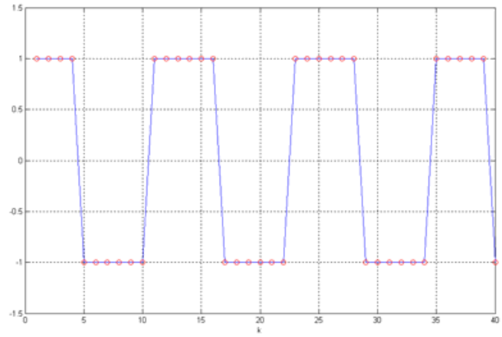

**Fig. 9** Switching variable $s$ when $h=1.5$

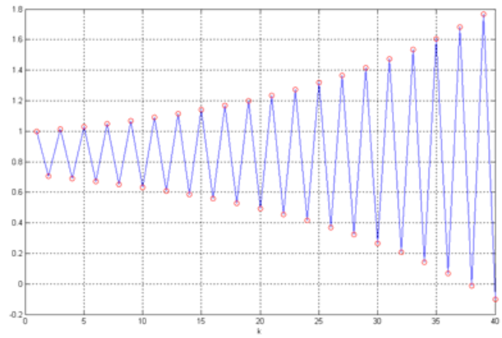

**Fig. 10** State variable $x_1$ when $h=0.9$

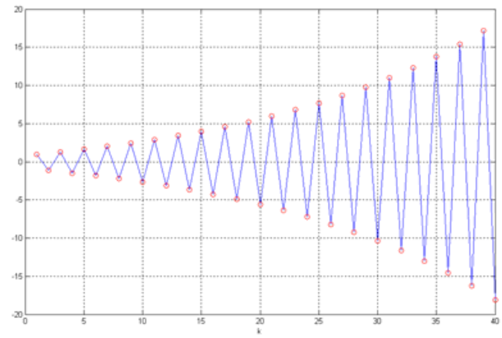

**Fig. 11** State variable $x_2$ when $h=0.9$

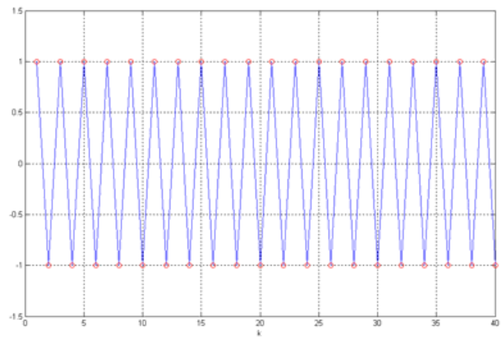

**Fig. 12** Switching variable $s$ when $h=0.9$