Iterative Learning Control for Linear Time-Variant Continuous Systems Based on Two-Dimensional System Theory *

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Abstract

In this paper, an iterative learning control (ILC) scheme is presented for linear time-variant continuous multi-variable systems based on two-dimensional (2-D) system theory. Three ILC schemes are discussed, and the corresponding convergence and effectiveness are proved where only the structure of 2-D system model, the property of λ-norm, and the Bellman-Gronwall inequality are employed. Two numerical simulation examples are included to validate the effectiveness of the proposed ILC procedures.

1. Introduction and Problem Formulation

Iterative learning control (ILC) was firstly introduced in 1984 by Arimoto et al. [1], and it has generated considerable research interest over the past years. The objective of ILC is to determine a control input iteratively, resulting in the plant's ability to track the given reference signal or the output trajectory over a fixed time interval. Hence, the most widely used ILC scheme is the PID-type scheme because this enables the conventional PID-like system for processing the tracking error [1]-[4]. However in [5], Geng et al. pointed that all PID-type ILC schemes inevitably suffer from a tight restriction. Moreover, the understanding of the structure and parameters of the unknown systems cannot be directly increased through the PID-type learning scheme because it is difficult to generalize the obtained results from a particular task to other similar tasks [6].

In fact, one of the main difficulties that ILC suffers is to establish a suitable mathematical model to clearly describe the dynamics of the control system and the behavior of the learning process [5]-[7]. Two-dimensional (2-D) model provides an excellent mathematical platform due to its two independent dynamic process [7]-[8], and hence 2-D system theory is introduced to ILC schemes.

In this paper, we discuss ILC problem for the following linear time-variant continuous system:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{1}$$

where $x(t) \in R^n$, $u(t) \in R^m$, $y(t) \in R^p$ are the state vector, the input vector, the output vector, respectively, and $A(t), B(t), C(t)$ are real time-variant matrices of appropriate dimensions that can be estimated. The boundary condition is $x(0) = x_0$.

Then, the ILC problem we are dealing with is stated as follows. Given system (1) with boundary condition $x(0) = x_0$, iteratively find an appropriate control input $\{u(t), 0 \le t \le T\}$ such that the system output follows the reference trajectory $y_d(t) \in R^p, 0 < t \le T$, i.e.,

$$\sup_{0<t\le T} \|y(t) - y_d(t)\| < \varepsilon$$

where $\varepsilon > 0$ is a required tolerance, and $y_d(0) = C(0)x_0$. Since the system matrices are not fully known, we are required to derive an ILC technique. In this paper, 2-D system theory is used to solve the above-mentioned problem. Main difficulty we solve is to establish a suitable 2-D system model to describe the dynamics of the control system and the behavior of the learning process, and a 2-D continuous-discrete Roesser's model is successfully derived by using the derivative of output tracking error instead of direct output tracking error.

2. ILC Schemes for Linear Time-Variant Continuous Systems

Suppose that k denotes the learning iteration, then a general ILC scheme is given as $u(t, k+1) = u(t, k) +$

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$\Delta u(t,k)$. Sequentially system (1) can be modeled as the following 2-D time-variant form

$$\begin{aligned} \frac{\partial x(t,k)}{\partial t} &= A(t)x(t,k) + B(t)u(t,k) \\ y(t,k) &= C(t)x(t,k) \end{aligned} \tag{2}$$

The boundary conditions for system (2) are given as $x(0,k) = x_0$ for $k = 0,1,2,\cdots$, and $u(t,0) = u_0(t)$ for $t \in [0,T]$. If $y_d(t)$ and $C(t)$ are differentiable for $t \in [0,T]$, denote

$$\begin{aligned} e(t,k) &= y_d(t) - y(t,k) \\ \xi(t,k) &= \frac{\partial e(t,k)}{\partial t} \\ \eta(t,k) &= x(t,k+1) - x(t,k) \end{aligned} \tag{3}$$

Thus, we can obtain

$$\begin{aligned} \frac{\partial \eta(t,k)}{\partial t} &= \frac{\partial (x(t,k+1) - x(t,k))}{\partial t} \\ &= A(t)\eta(t,k) + B(t)\Delta u(t,k) \end{aligned} \tag{4}$$

$$\begin{aligned} \xi(t,k+1) - \xi(t,k) &= \frac{\partial (e(t,k+1) - e(t,k))}{\partial t} \\ &= -\frac{\partial (y(t,k+1) - y(t,k))}{\partial t} \\ &= -[C(t)A(t) + \dot{C}(t)]\eta(t,k) \\ &\quad -C(t)B(t)\Delta u(t,k) \end{aligned} \tag{5}$$

Equations (4) and (5) can be written in compact form

$$\begin{bmatrix} \frac{\partial \eta(t,k)}{\partial t} \\ \xi(t,k+1) \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ -C(t)A(t) - \dot{C}(t) & I \end{bmatrix} \begin{bmatrix} \eta(t,k) \\ \xi(t,k) \end{bmatrix} + \begin{bmatrix} B(t) \\ -C(t)B(t) \end{bmatrix} \Delta u(t,k) \tag{6}$$

Applying the following ILC scheme

$$\Delta u(t,k) = K(t)\xi(t,k) \tag{7}$$

we can derive a system with respect to the derivative of control error in accordance with the 2-D continuous-discrete Roesser's model

$$\begin{aligned} &\begin{bmatrix} \frac{\partial \eta(t,k)}{\partial t} \\ \xi(t,k+1) \end{bmatrix} \\ &= \begin{bmatrix} A(t) & B(t)K(t) \\ -C(t)A(t) - \dot{C}(t) & I - C(t)B(t)K(t) \end{bmatrix} \begin{bmatrix} \eta(t,k) \\ \xi(t,k) \end{bmatrix} \end{aligned} \tag{8}$$

The boundary conditions for (8) are $\eta(0,k) = 0$ for $k = 0,1,2,\cdots$ and finite $\xi(t,0)$ for $t \in [0,T]$. Also, suppose that there exists a positive number $L_{T_{10}}$ such that the following inequality holds:

$$\left\| \begin{bmatrix} A(t) & B(t)K(t) \\ 0 & 0 \end{bmatrix} \right\| \leq L_{T_{10}} \quad for\ t \in [0,T] \tag{9}$$

Thus, the following theorem can be proved by employing the structure of 2-D system model, the property of λ-norm, and the Bellman-Gronwall inequality.

Theorem 1. *For a 2-D ILC model (2), suppose that both the desired output $y_d(t)$ and system matrix $C(t)$ are differentiable for $t \in [0,T]$. If*

$$\sup_{0 \leq t \leq T} \left\| \begin{bmatrix} 0 & 0 \\ -C(t)A(t) - \dot{C}(t) & I - C(t)B(t)K(t) \end{bmatrix} \right\| < 1 \tag{10}$$

then the ILC scheme

$$u(t,k+1) = u(t,k) + K(t)\left[\frac{dy_d(t)}{dt} - \frac{\partial y(t,k)}{\partial t}\right] \tag{11}$$

can ensure that $\lim_{k \to \infty} e(t,k) = 0$ for $t \in [0,T]$.

According to Theorem 1, algorithm 1 is introduced.

Algorithm 1:

1). The system matrices $A(t), B(t), C(t)$, the reference output trajectory $y_d(t)$, the required form $\phi(t)$ of the whole resulting error matrix $I - C(t)B(t)K(t)$, and the trajectory tolerance $\varepsilon > 0$ are given for $t \in [0,T]$.

2). Let $k = 0$, $u_0(t) = 0$, $x(0) = x_0$, $K(t) = (C(t)B(t))^T [C(t)B(t)(C(t)B(t))^T]^{-1} (I - \phi(t))$.

3). According to system (2), calculate $y(t,k)$. If $\sup_{0<t\leq T} \|y(t,k) - y_d(t)\| \geq \varepsilon$, then calculate $u(t,k+1)$ according to (11), else go to step 5).

4). $k = k+1$, return to step 3).

5). End.

To improve the learning efficiency, a modification of learning scheme (7) is given as: $\Delta u(t,k) = -K_1(t)\eta(t,k) + K_2(t)\xi(t,k)$. Then, the system (8) can be formulated as

$$\begin{bmatrix} \frac{\partial \eta(t,k)}{\partial t} \\ \xi(t,k+1) \end{bmatrix} = \begin{bmatrix} A_{11}(t) & A_{12}(t) \\ A_{21}(t) & A_{22}(t) \end{bmatrix} \begin{bmatrix} \eta(t,k) \\ \xi(t,k) \end{bmatrix} \tag{12}$$

where submatrices are given as $A_{11}(t) = A(t) - B(t)K_1(t)$, $A_{12}(t) = B(t)K_2(t)$, $A_{21}(t) = -C(t)A(t) - \dot{C}(t) + C(t)B(t)K_1(t)$, $A_{22}(t) = I - C(t)B(t)K_2(t)$. If $C(t)B(t)$ has uniform full-row rank for $t \in [0,T]$, then we have

$$\begin{aligned} \hat{K}_1(t) &= (C(t)B(t))^+ [C(t)A(t) + \dot{C}(t)] \\ \hat{K}_2(t) &= (C(t)B(t))^+ \end{aligned}$$

so that $-C(t)A(t) - \dot{C}(t) + C(t)B(t)\hat{K}_1(t) = 0$ and $I - C(t)B(t)\hat{K}_2(t) = 0$, where $(\cdot)^+$ represents the Moore-Penrose inverse of matrix. Set $K_1(t) = \hat{K}_1(t), K_2(t) = \hat{K}_2(t)$, and hence we have $\xi(t,1) = 0$ no matter what $\xi(t,0)$ is. Then, we can obtain $e(t,1) = 0$ for $t \in [0,T]$ based on the initial condition $y_d(0) = C(0)x_0$.

Theorem 2. *For a 2-D ILC model (2), if $C(t)B(t)$ has uniform full-row rank for $t \in [0,T]$, then there exists an*

ILC scheme

$$\begin{aligned} u(t,k+1) = u(t,k) - \hat{K}_1(t)[x(t,k+1) - x(t,k)] \\ + \hat{K}_2(t)\left[\frac{dy_d(t)}{dt} - \frac{\partial y(t,k)}{\partial t}\right] \end{aligned} \tag{13}$$

that can drive the control error to zero for the whole reference output trajectory after only one learning trial.

Though Theorem 2 provides an effective ILC scheme (13), $x(t,k+1)$ is not available, and hence further modification is needed. If system matrices of (1) are accurately known, then from equations (2) and (13), we can derive

$$\begin{aligned} \frac{\partial x(t,k+1)}{\partial t} &= [A(t) - B(t)\hat{K}_1(t)]x(t,k+1) \\ &+ B(t)[I - \hat{K}_2(t)C(t)B(t)]u(t,k) \\ &+ B(t)\hat{K}_1(t)x(t,k) + B(t)\hat{K}_2(t)\times \\ &\left[\frac{dy_d(t)}{dt} - (C(t)A(t) + \dot{C}(t))x(t,k)\right] \end{aligned} \tag{14}$$

Therefore, we can apply control law

$$\begin{aligned} \hat{u}(t) = [I - \hat{K}_2(t)C(t)B(t)]u(t) + \hat{K}_1(t)x(t) \\ + \hat{K}_2(t)\left[\frac{dy_d(t)}{dt} - (C(t)A(t) + \dot{C}(t))x(t)\right] \end{aligned} \tag{15}$$

to the following system

$$\begin{aligned} \dot{x}(t) &= [A(t) - B(t)\hat{K}_1(t)]x(t) + B(t)\hat{u}(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{16}$$

which is the state feedback form of system (1). Hence, the output of the closed-loop system is identical with the reference output, namely, $y(t) = y_d(t), t \in [0,T]$. This result can be directly verified by the response formula of system (16). Thus, the following theorem can be proved.

Theorem 3. *For a 2-D ILC model (2), if the matrix $C(t)B(t)$ has uniform full-row rank for $t \in [0,T]$, then the following ILC scheme*

$$u(t) \Leftarrow \hat{u}(t) - \hat{K}_1(t)\hat{x}(t) \tag{17}$$

can drive the control error to zero for the whole reference output trajectory after only one learning iteration, where $\hat{x}(t)$ is the state vector of system (16).

Similarly, we give the second algorithm based on Theorem 3.

Algorithm 2:

1). For $t \in [0,T]$, the system matrices $A(t), B(t), C(t)$, the reference output trajectory $y_d(t)$, the initial input sequence $u(t)$, and the initial state vector of system $x(0) = x_0$ are given.

2). Calculate $\hat{K}_1(t), \hat{K}_2(t)$, and measure $x(t), y(t)$ from system (1).

3). Use (15) to calculate $\hat{u}(t)$, then apply $\hat{u}(t)$ to system (16) and measure $\hat{x}(t)$.

4). Apply control $\hat{u}(t) - \hat{K}_1(t)\hat{x}(t)$ to system (1).

3. Examples

Example 1: Consider the ILC problem for the following linear time-variant continuous system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} -0.1cos(t^{0.2}) & 3 \\ 0.02t & 10sin(t) \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0.027t + 1 \\ 0.12 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0.45 & -0.001t \end{bmatrix} x(t) \end{aligned} \tag{18}$$

where $x(0) = [0\ 0]^T$, and the matrix $C(t)B(t)$ has uniform full-row rank for $t \in [0,1]$. The desired output is given as $y_d(t) = 12t^2(1-t)$. Using Algorithm 1, we set the initial input sequence of ILC as $u(t,0) = 0$, and let $K(t) = 0.6(C(t)B(t))^T[C(t)B(t)(C(t)B(t))^T]^{-1}$. Additionally, the accuracy of tracking is evaluated by the following total square error of tracking:

$$S = \int_0^1 [y_d(\tau) - y(\tau)]^2 d\tau$$

Figure 1 shows the tracking error performance of the ILC system output at different time-steps and iterations, and Figure 2 performs the curves of the total square error of tracking in the process of Algorithm 1 being iteratively executed. From Figure 1-2, it can be concluded that the convergence rate of Algorithm 1 is high and the output is capable of tracking the desired trajectory accurately within few iterations.

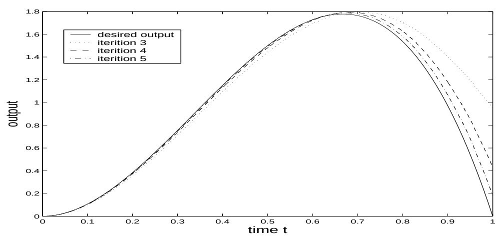

Figure 1: (Example 1) Tracking error performance of ILC system output using Algorithm 1.

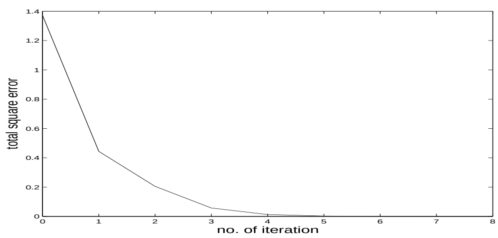

Figure 2: (Example 1) Total square error of different iterations using Algorithm 1.

Example 2: To demonstrate Algorithm 2, consider the following linear time-variant continuous system:

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0.18 & 0 \\ 0.02t & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 0.1 \\ 0.01t+2 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -0.52 & 0 \end{bmatrix} x(t) \end{aligned} \tag{19}$$

where $x(0) = [0\,0]^T$, and the matrix $C(t)B(t)$ has uniform full-row rank for $t \in [0,1]$. The desired output is given as $y_d(t) = sin(\pi t)$. Here, provided that the accurate information on the system parameters in system (19) is unavailable, and only its estimated system is given as

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0.2 & 0 \\ 0.02t & -0.46 \end{bmatrix} x(t) + \begin{bmatrix} 0.13 \\ 0.015t+2.1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} -0.54 & -0.01 \end{bmatrix} x(t) \end{aligned} \tag{20}$$

Despite of this situation, Algorithm 2 is still effective. Figure 3 shows the tracking performance of the ILC system output at different time-steps and iterations. Also, Figure 4 describes the curves of total square error of tracking in the process of Algorithm 2 being iteratively executed. From Figure 3-4, it can be noticed that it takes few iterations for Algorithm 2 to drive the tracking error to a very low level for the whole desired output. Moreover, this simulation result demonstrates that Algorithm 2 is robust with respect to small perturbations of system parameters.

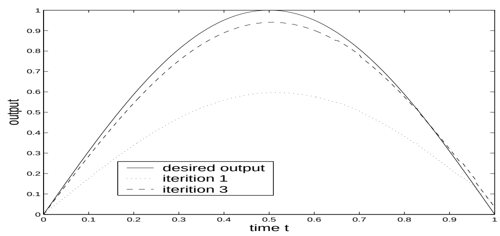

Figure 3: (Example 2) Tracking error performance of the system output using Algorithm 2.

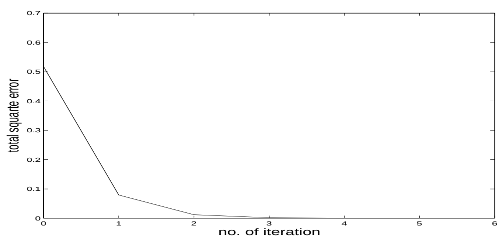

Figure 4: (Example 2) Total square error of different iterations using Algorithm 2.

4. Conclusions

Main difficulty of ILC is to establish a suitable mathematical model to describe the dynamics of the control system and the behavior of the learning process. This paper successfully established the 2-D continuous-discrete Roesser's model with respect to the derivative of the output tracking error. Then, three ILC schemes were given, and sufficient conditions for convergence of these three learning schemes were presented. Example 1-2 validated that these ILC procedures were effective and robust with respect to small perturbations of system parameters.

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