Analysis of bifurcation and optimal response on the evolution of cooperation

Yo-Hey Otake
Graduate School of Frontier Sciences
The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
yoheyo@sat.t.u-tokyo.ac.jp

Kazuyuki Aihara
Institute of Industrial Science
The University of Tokyo
4-6-1 Komaba, Meguro-ku, Tokyo 153-8505, Japan
and Aihara Complexity Modelling Project, ERATO, JST
45-18 Oyama-cho, Shibuya-ku, Tokyo 151-0065, Japan
aihara@sat.t.u-tokyo.ac.jp

Abstract

A variety of strategies are needed to maintain cooperative behavior. Such strategy diversity in replication arises from various circumstances; for example, mutation in replication, noise, mistakes and moods. In this paper, we deal with the iterated prisoner's dilemma game, which has been widely used to study the evolution of cooperation. We approach the question of how cooperation evolves from the standpoint of dynamical systems and also analyze the evolution in terms of optimal response. Through these analyses, we have confirmed that strategy variation is important for the evolution of cooperation. In addition, we show that this approach is more useful than previous approaches because use of dynamical systems theory allows us to express a transient process dynamically.
Keywords: evolutionary game dynamics, replicator equation, cooperative behavior, iterated prisoner's dilemma, bifurcation analysis, optimal response.

1 Introduction

1.1 Evolutionary game theory

An aim of sociology and economics is to understand how cooperative behavior is maintained. It is important for research on behavior to look at the sustainability of such phenomena as well as their initial causes.

For this purpose, Maynard Smith applied game theory to ecological situations[1], which is known as evolutionary game theory. The main concept of this approach is that an evolutionarily stable strategy can be achieved. This concept of stability enables us to discuss sustainability.

Evolutionary game theory is now widely applied to many fields such as behavioral biology, ecology and economics.

1.2 Need for dynamic analysis

Most evolutionary researches on cooperation have emphasized an invasion condition of cooperative behavior in a population characterized by selfish behavior. To make up for the insufficiency of this approach, we need mathematical tools for analyzing dynamic processes as well as static states. Therefore, we have used nonlinear dynamical systems analysis to analyze the global behavior of evolutionary dynamics in the phase space of a strategy set.

1.3 Analysis of mutation in the evolution

Past researches have not accounted for all properties of evolution. Our goal is thus to consider situations containing various evolutionary factors, especially effects of mutation.

Through consideration of mutation, we examine effects of diversity on the evolution of cooperation. For this analysis, we emphasize the importance of dynamical systems, especially through bifurcation analysis.

2 Theory and methods

First, we introduce the game used to model the situations we consider, namely the prisoner's dilemma game, and prepare a strategy set for playing the game. After that, we introduce tools for the analysis. We describe the replicator equation and the form of mutation.

2.1 The Iterated Prisoner's Dilemma game

The iterated prisoner's dilemma (IPD)[1] is the most widely used model for the evolution of cooperation. Axelrod held a competition to study IPD strategy through a computer simulation [2]. This competition showed that a "tit-for-tat" (TFT) strategy was the most advantageous. A player using this strategy cooperates in the first interaction and then in subsequent interactions repeats (imitates) what the opponent did in the immediately preceding interaction.

On the other hand, the evolutionarily stable strategy (ESS) [1] in a finite IPD is the "always defect" (AllD) strategy. A violation of AllD, which is equivalent to destabilization of the ESS, is necessary for the evolution of cooperation. The evolution of cooperation in such cases can arise from various conditions (for example, groupings or spatial structures).

2.2 Strategy set

We must consider a class of various types of strategy ranging from selfish behavior to cooperative behavior. Thus, we prepared a strategy set (Table 1) like those used elsewhere [3, 4, 5].

strategy	explanation
TFT	Tit for tat
E1	Tit for tat and defect in the last interaction
E2	Tit for tat and defect in last two interactions
⋮	⋮
Ek	AllD (The number of interactions is k)

Table 1: Strategy set

TFT represents cooperative behavior (not being the first to defect) and AllD represents selfish behavior. This strategy set is filled from cooperation to defection.

[1]A payoff matrix for the prisoner's dilemma game (PD) is shown below.

		Opponent	
		Cooperation	Defection
Self	Cooperation	R	S
	Defection	T	P

S < P < R < T and T + S < 2R. This situation leads to the self's conflict and dilemma: if each of two players chooses the behavior maximizing self payoff, it brings about the situation minimizing the sum of the payoffs. Through iteration of the PD, the tendency towards cooperation increases.

2.3 Evolutionary game dynamics: the replicator equation

The fundamental law of evolutionary game theory is described by the replicator equation [6], and the evolutionary path can be understood as the dynamics on the phase space spanned by the frequency of each strategy.

We consider a replicator map of the following form:

$$x_i(t+1) = F_i(\vec{x}(t)) = \frac{x_i(t)w_i(\vec{x}(t))}{\sum_{j=1}^{N} x_j(t)w_j(\vec{x}(t))}, \quad (1)$$

where the variable x_i denotes the frequency of strategy i, which fitness, $w_i(\vec{x})$, is a function of the distribution of the population given by the vector $\vec{x} = (x_1, \cdots, x_n)$. The denominator, $\sum_{j=1}^{N} x_j(t)w_j(\vec{x}(t))$, ensures that $\sum_{j=1}^{N} x_j(t) = 1$. This map describes frequency-dependent selection. The evolutionary game theory assumes that Darwinian fitness is determined by the payoff matrix of a game (e.g., the IPD).

If the number of iterations of an IPD is finite and fixed[2], this map is deterministic and has no stochasticity.

We carried out numerical simulations based on this map. Similar analyses of population dynamics based on game theory have been done in [7].

2.4 Mutation

The elements of the evolution are as follows:
- Heredity
- Selection
- Mutation.

The replicator system mentioned above does not allow mutation, then we introduce the effect of mutation.

In this work, we discuss the replicator-mutator map of the following form [8, 9, 10]:

$$x_i(t+1) = F_i(\vec{x}(t)) = \frac{\sum_{j=1}^{N} x_j(t)w_j(\vec{x}(t))q_{ji}}{\sum_{j=1}^{N} x_j(t)w_j(\vec{x}(t))}, \quad (2)$$

where each setting is the same as for the replicator map (1). The probability that replication of strategy j gives rise to strategy i is given by q_{ji}. These quantities define the mutation matrix (a Markov matrix). This map describes both frequency-dependent selection and

[2]As in previous studies [3, 4, 5].

mutation.

Here, we consider a mutation matrix (Markov matrix) with uniform mutation[3] of the following form:

$$(q_{ji}) = \begin{cases} 1-(N-1)\epsilon & (j=i) \\ \epsilon & (j \neq i) \end{cases} . \qquad (3)$$

We consider the population dynamics of this model and analyze the stability and bifurcations of this system.

3 Results

In this section, we analyze the effects of mutation. First, we show time series obtained by numerical simulation with several mutation rates. After that, we show a bifurcation diagram and discuss its bifurcation structure. The number of strategies is N and the number of iterations of the IPD is k (N=k+1).

3.1 No mutation: the replicator equation

Previous studies [3, 4, 5] considered the case of no mutation. No mutation is equivalent to the mutation matrix being an identity matrix, namely $\epsilon = 0$ in Equation (3).

A numerical simulation result of the equation is shown in Figure 1.

The orbit in the simulation resembles a heteroclinic cycle, and approaches each of the corners on the k-dimensional simplex.

The sojourn time in each strategy exponentially increases. Various researchers have discussed the possibility that because the evolution of non-cooperative strategies cause the waste of much time, cooperation can evolve.

[3]We can also consider other types of formalization of the mutation matrix. A simple model is

$$(q_{ji}) = \begin{cases} 1-2\epsilon & (j=i) \quad (1-\epsilon \text{ if } j=1 \text{ or } N) \\ \epsilon & (j=i-1, i+1) \end{cases} .$$

A strategy easily mutates into a similar strategy on the phenotype, but cannot mutate into a radically different strategy. The distance on the phenotype is strictly determined by the distance on the genotype. In this model, mutation will similarly affect the evolution of cooperation, but various dynamics would not be able to arise.

3.2 Slight mutation

Next, we investigated effects of mutation.

While the orbit property is the same as in the case of no mutation, the population dynamics are drastically changed even by a slight mutation.

A numerical simulation result of evolutionary dynamics with a slight mutation is shown in Figure 2.

If there is a slight mutation, the sojourn time in each strategy except the final one (AllD) is constant. Therefore, the evolution time of non-cooperative behavior is linear and selfish behavior evolves in actual time. Since the models used in previous studies cannot tolerate even a small amount of mutation, we consider them insufficient for interpreting the evolution of cooperation.

3.3 Greater mutation

Since there are reasons other than those considered in previous studies that can account for the evolution of cooperation, we look at the conditions affecting the evolution.

One factor that helps explain the evolution of cooperation is effective mutation.

We make the variation of strategy wider. In addition, the population dynamics change drastically depending on the mutation rate. In this instance, moreover, the orbit property differs from those in the cases of no mutation and a slight mutation.

A numerical simulation result of the evolutionary dynamics with a fairly weak mutation is shown in Figure 3.

For any initial condition, the dynamics fall into this quasi-periodic orbit. Thus, non-cooperative behavior cannot evolve. The result of evolution is periodic change of behavior.

3.4 Bifurcation analysis

We measure the level of cooperation in the population as a function of the mutation parameter ϵ. This is the level of cooperation after sufficient time has passed and the influence of the initial conditions is negligible. The bifurcation diagram is shown in Figure 4.

A mutation threshold exists and the dynamics of this system drastically change with changing the value of ϵ. The period of the limit cycle also continuously changes.

This system contains a saddle-node bifurcation on an invariant circle and a Neimark-Sacker bifurcation [11].

4 Discussion

We assert that mutation may play an important role in the evolution of cooperation, because population dynamics are drastically changed by mutation in replication. Here, we discuss the bifurcation at this change.

4.1 Comparison with recent research

Recent research based on dynamic programming suggested similar outcomes [12]. Therefore, we compare a dynamical systems technique to an analysis of optimal response and discuss the similarities and differences between the former research and our own.

The earlier research on the evolution of cooperation through analysis of optimal response has two limitations. First, a limited situation was assumed where most of the population adopted one strategy and other strategies were distributed among the population according to a distribution of a given variance. If a state is a mixture of strategies, it is difficult to analyze. Second, the response could only be examined for certain static situations, and dynamic processes could not be accommodated. McNamara et al. [12] demonstrated the importance of mutation (variation) from this viewpoint.

To determine whether cooperative behavior is sustainable, we need to describe the time evolution of behavior. An approach based on a dynamical system allows the expression of various states like mixtures of strategies and transient processes. The resulting abundance of information makes this a more fruitful approach. This is important for studying the evolution of cooperation [13].

The advantage of the approach with a dynamical system is that it allows us to examine not only the direction of evolution but also the time evolution of the frequencies of the strategies and transient processes. This is what makes the dynamical system approach so effective.

4.2 Consideration as game theory

In this paper, we have dealt with a finitely iterated prisoner 's dilemma game; that is, we have discussed the situation where the period of interaction is fixed and known. This is a difficult problem in the sense that it is difficult for cooperative behavior to evolve.

In biological and economic situations, the period of interaction may be known to everyone. In this case, cooperative behavior is maintained. Nevertheless, it is difficult for cooperation to evolve theoretically because of backward induction reasoning, and few theoretical approaches have allowed the evolution of cooperation.

When the period of interaction is fixed and known, previous studies have shown that it is difficult for cooperation to evolve because of backward induction reasoning [3, 4, 5]. Backward induction is powerful premise, leading some to consider theoretical research on the evolution of cooperation to be impossible. The few studies that supported the evolution of cooperation [3, 4, 5, 14] used models that did not contain all properties of evolution. Therefore, we wanted to consider situations involving various evolutionary factors, especially the effect of mutation.

Our goal is to bridge the gap between the theoretical and empirical researches. In this paper, we sought to do this by incorporating the role of mutation. By considering mutation, we found a remarkable phenomenon: mutation can promote the evolution of cooperation. This phenomenon arises from a bifurcation of the equation solution.

5 Conclusion

We have examined the factors affecting the evolution of cooperation in a finitely IPD. Players cooperate in a situation where the interaction term is finite and known. This evolution is supported by mutation. Our research indicates that mutation plays an important role in the evolution of cooperation. Moreover, through a comparison with another approach, we have confirmed the effectiveness of applying dynamical systems theory to the evolution of behavior.

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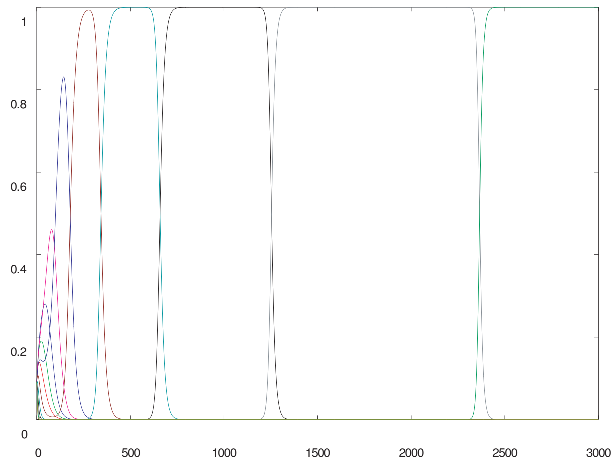

Figure 1: Population dynamics without mutation (k = 10, $\epsilon = 0$). The horizontal axis is time (generation) and the vertical axis is the population frequency of each strategy.

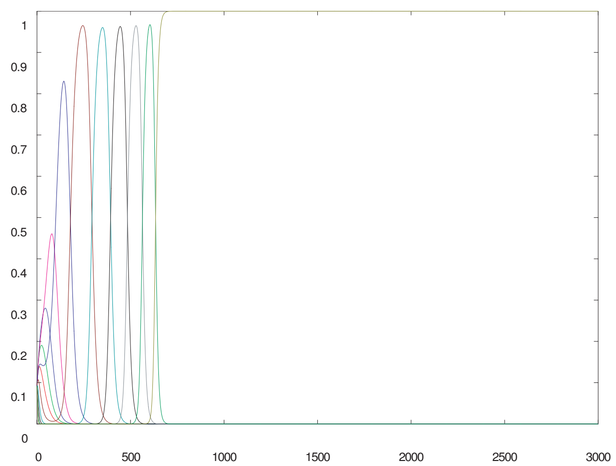

Figure 2: Population dynamics with a slight mutation (k = 10, $\epsilon = 1.0 \times 10^{-6}$). The horizontal axis is time (generation) and the vertical axis is the population frequency of each strategy.

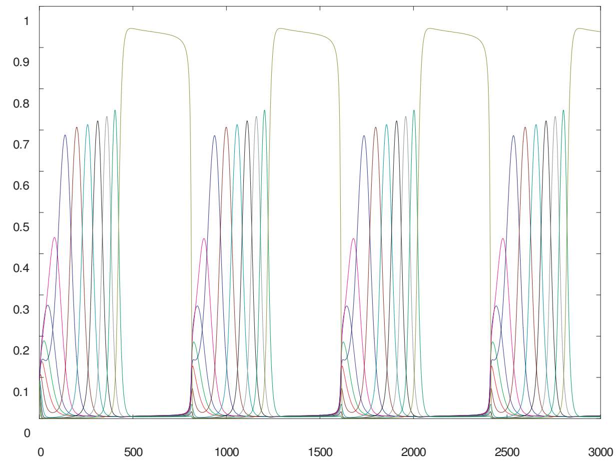

Figure 3: Population dynamics with a weak mutation (k = 10, $\epsilon = 3.5 \times 10^{-4}$). The horizontal axis is time (generation) and the vertical axis is the population frequency of each strategy.

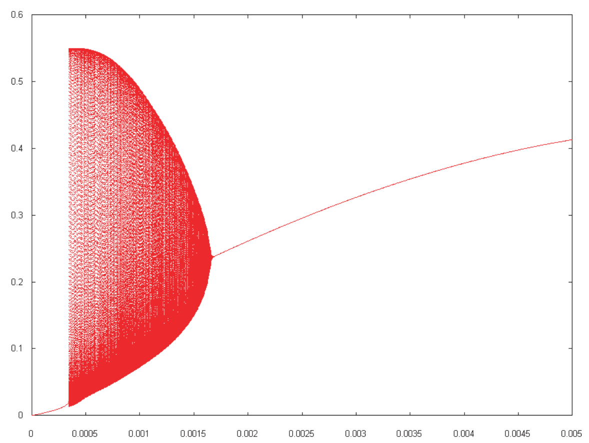

Figure 4: Bifurcation diagram: The horizontal axis is the mutation rate (the bifurcation parameter) and the vertical axis is the frequency of cooperative behavior after sufficient time has passed.