Some Properties of Four-Dimensional Multicounter Automata

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Abstract

Recently, due to the advances in many application areas such as computer animation, motion image processing, and so forth, it has become increasingly apparent that the study of four-dimensional pattern processing has been of crucial importance. Thus, we think that the research of four-dimansional automata as a computational model of four-dimensional pattern processing has also been meaningful. This paper introduces four-dimensional multicounter automata, and investigates some their properties. We show the differences between the accepting powers of seven-way and eight-way four-dimensional multicounter automata, and between the accepting powers of deterministic and nondeterministic seven-way four-dimensional multicounter automata.

Key Words **:** computational complexity, four-dimensional automaton, multicounter, nondeterminism.

1 Introduction and Preliminaries

Inoue et al. [5] introduced a two-dimensional multicounter automaton and investigated its basic properties. Szepietowski also investigated some of its properties [10]. A two-dimensional k-counter automaton M is a two-dimensional finite automaton [1] that has k counters. The action of M is similar to that of the one-dimensional off-line k-counter machine [3], except that the input head of M can move up, down, right, or left on a two-dimensional input tape. In [7], Sakamoto et al. introduced multicounter automata on three-dimensional input tapes.

By the way, during the past about forty years, several automata on two- or three-dimensional tapes have been proposed and many properties of them have been obtained [6,8]. On the other hand, recently, due to the advances in computer animation, motion image processing, and so on, the study of four-dimensional information processing has been of great importance. Thus, we think that the study of four-dimensional automata has been meaningful as the computational model of four-dimensional information processing [9].

In this paper, we introduce and investigate about eight-way four-dimensional multicounter automata as new four-dimensional computational models. An eight-way four-dimensional k-counter automaton (4-kCA), which consists of a finite control, k counters, a read-only four-dimensional input tape, k counter heads, and an input head which can move in eight directions — north, east, south, west, up, down, future or past. In general, when we must think about the algorithm of four-dimensional pattern processing by using the restricted computational resources, if the algorithm is fine in spite of its restricted computational resources, it will be valued highly. It is the same with automata theory. So we next introduce and investigate a seven-way four-dimensional k-counter automaton ($SV4$-kCA) which is a restricted type of 4-kCA. $SV4$-kCA is a 4-kCA whose input head can move in seven directions — north, east, south, west, up, down, or future. In this paper, we let each sidelength of each input tape of these automata be equivalent in order to increase the theoretical interest.

Let Σ be a finite set of symbols. *A four-dimensional tape* over Σ is a four-dimensional rectangular array of elements of Σ. The set of all the four-dimensional tapes over Σ is denoted by $\Sigma^{(4)}$.

Given a tape $x \in \Sigma^{(4)}$, for each $j (1 \leq j \leq 4)$, we let $l_j(x)$ be the length of x along the jth axis. The set of all $x \in \Sigma^{(4)}$ with $l_1(x) = m_1$, $l_2(x) = m_2$, $l_3(x) = m_3$, and $l_4(x) = m_4$ is denoted by $\Sigma^{(m_1,m_2,m_3,m_4)}$. When $1 \leq j_i \leq l_j(x)$ for each $j (1 \leq j \leq 4)$, let $x(i_1, i_2, i_3, i_4)$ denote the symbol in x with coordinates (i_1, i_2, i_3, i_4). Furthermore, we define

$$x[(i_1, i_2, i_3, i_4), (i'_1, i'_2, i'_3, i'_4)],$$

when $1 \leq i_j \leq i'_j \leq l_j(x)$ for each integer $j (1 \leq j \leq 4)$, as the four-dimensional tape y satisfying the following (i) and (ii) :

(i) for each $j (1 \leq j \leq 4)$, $l_j(y) = i'_j - i_j + 1$;

(ii) for each r_1, r_2, r_3, r_4 $(1 \leq r_1 \leq l_1(y), 1 \leq r_2 \leq l_2(y), 1 \leq r_3 \leq l_3(y), 1 \leq r_4 \leq l_4(y))$, $y(r_1, r_2, r_3, r_4) = x(r_1 + i_1 - 1, r_2 + i_2 - 1, r_3 + i_3 - 1, r_4 + i_4 - 1)$.

Four-dimensional tape is the sequence of three-dimensional rectangular arrays along the time axis. By $Cube_x(i)$ $(i \leq 1)$, we denote the ith three-dimensional rectangular array along the time axis in a tape $x \in \Sigma^{(4)}$ which each sidelength is equivalent.

We now introduce a seven- or eight-way four-dimensional multicounter automaton. An *eight-way four-dimensional k-counter automaton* (4-kCA) M, $k \geq 1$, has a read-only four-dimensional input tape surrounded by *boundary symbols* $\sharp$'s and k *counters.* (Of course, M has a *finite control*, an *input head*, and k *counter heads.*) The action of M is similar to that of the two- or three-dimensional multicounter automaton [5, 7, 10], except that the input head of M can move in eight directions — east, west, south, north, up, down, future, or past. That is, when an input tape $x \in \Sigma^{(4)}$ (where Σ is the set of input symbols of M and the boundary symbol $\sharp$'s is not in Σ) is presented to M, M determines the next state of the finite control, the move direction (east, west, south, north, up, down, future, past, or no move) of the input head, and the move direction (right, left, or no move) of each counter head, depending on the present state of the finite control, the symbol read by the input head, and whether or not the contents of each counter is zero (i.e., whether or not each counter head is on the bottom symbol Z_0 of the counter). If the input head falls off the tape x with boundary symbols, M can make no further move. M starts in its initial state, with the input head on position (1,1,1,1) of the tape x, and with the contents of each counter zero (i.e., with each counter on the bottom symbol Z_0 of the counter). We say that M *accepts* the tape x if M eventually halts in a specified state (accepting state) on the bottom boundary symbol $\sharp$ of the input. We denote by $T(M)$ the set of all the four-dimensional tapes accepted by M. A *seven-way four-dimensional k-counter automaton* ($SV4$-kCA) is a 4-kCA whose input head can move in seven directions — east, west, south, north, up, down, or future (see Fig.1).

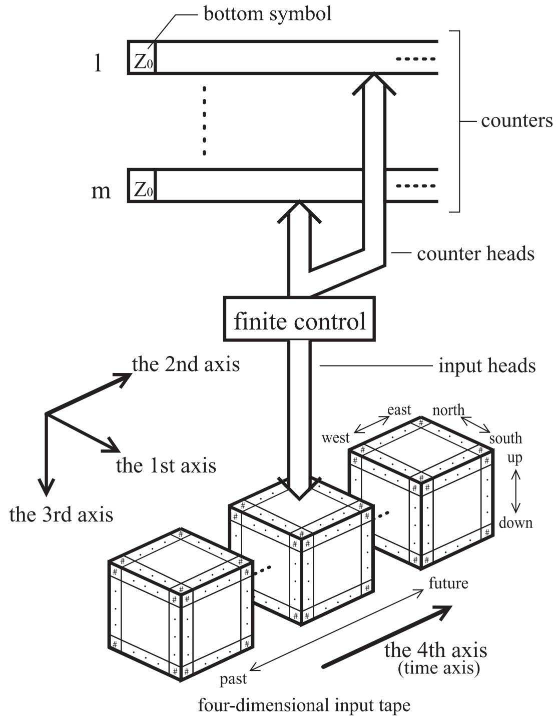

Fig. 1: Four-dimensional k-counter automaton.

Let $L(m) : \mathbf{N} \mapsto \mathbf{N}$ (where $\mathbf{N}$ is the set of all the positive integers) be a function with one variable m. A 4-kCA ($SV4$-kCA) M is said to be $L(m)$ *counter-bounded* if for each $m \geq 1$ and each input tape x (accepted by M) with $l_1(x) = l_2(x) = l_3(x) = l_4(x) = m$, the length of each counter of M is bounded by $L(m)$. As usual, we define nondeterministic and deterministic 4-kCA's ($SV4$-kCA's). By $N4$-$kCA(L(m))$ (respectively, $D4$-$kCA(L(m))$, $NSV4$-$kCA(L(m))$, and $DSV4$-$kCA(L(m))$), we denote a nondeterministic 4-kCA (respectively, deterministic 4-kCA, nondeterministic $SV4$-kCA, and deterministic $SV4$-kCA) whose each sidelength of each input tape is equivalent and which is $L(m)$ counter-bounded. Let $\mathcal{L}[N4\text{-}kCA(L(m))] = \{T \mid T = T(M)$ *for some* $N4\text{-}kCA(L(m))\ M\}$. $\mathcal{L}[N4\text{-}kCA(L(m))]$, $\mathcal{L}[NSV4\text{-}kCA(L(m))]$, and $\mathcal{L}[DSV4\text{-}kCA(L(m))]$ have similar meanings.

We briefly recall *seven-way four-dimensional Turing machines* [9]. A seven-way four-dimensional Turing machine M has a read-only four-dimensional input tape with boundary symbols $\sharp's$ and one semiinfinite *storage tape.* (Of course, M has a *finite control*, an *input head*, and a *storage-tape head.*) The action of M is similar to that of the two- or three-dimensional Turing machine [6,8] which has a read-only input tape with boundary symbols $\sharp's$ and one semiinfinite storage tape, except that the input head of M can move in seven directions — east, west, south, north, up, down, or future. M starts in its initial state, with the input head on position $(1, 1, 1, 1)$ of an input tape x, and with all the cells of the storage tape blank. We say that M *accepts* the tape x if M eventually halts in an accepting state. Let $L(m) : \mathbf{N} \mapsto \mathbf{N}$ be a function with one variable m. By $NSV4$-$TM(L(m))$ ($DSV4$-$TM(L(m))$) we denote a nondeterministic (deterministic) seven-way four-dimensional Turing machine whose each sidelength of each input tape is equivalent and which does not scan more than $L(m)$ cells on the storage tape for any input tape x (accepted by M) with $l_1(x) = l_2(x) = l_3(x) = l_4(x) = m$. Let $\mathcal{L}[NSV4\text{-}TM(L(m))]$ ($\mathcal{L}[DSV4\text{-}TM(L(m))]$) denote the class of sets accepted by $NSV4$-$TM(L(m))$'s ($DSV4$-$TM(L(M))$'s).

We denote a nondeterministic (deterministic) four-dimensional finite automaton by $N4$-FA ($D4$-FA). A seven-way $N4$-FA (seven-way $D4$-FA) is an $N4$-FA ($D4$-FA) whose input tape head can move in seven directions — east, west, south, north, up, down, or future. By $N4$-FA ($D4$-FA, $NSV4$-FA, $DSV4$-FA) we denote an $N4$-FA ($D4$-FA, seven-way $N4$-FA, seven-way $D4$-FA) whose each sidelength of each input tape is equivalent [9]. For example, let $\mathcal{L}[D4\text{-}FA]$ denote the class of sets accepted by $D4$-FA's. As is easily seen, it follows that for any constant k, $\mathcal{L}[D4\text{-}FA] = \mathcal{L}[D4\text{-}1CA(k)]$, $\mathcal{L}[DSV4\text{-}FA] = \mathcal{L}[DSV4\text{-}1CA(k)]$, and so on.

We conclude this section by giving a relationship between seven-way four-dimensional multicounter automata and seven-way four-dimensional Turing machines, which will be used in the latter sections.

[Theorem 1.1]

(1) $\bigcup_{1\le k<\infty} \mathcal{L}[XSV4\text{-}kCA(L(m))] \subseteq \mathcal{L}[XSV4\text{-}TM(\log L(m))]$ *for any* $L(m) : \mathbf{N} \mapsto \mathbf{N}$ *and any* $X \in \{D, N\}$,

(2) $\bigcup_{1\le k<\infty} \mathcal{L}[XSV4\text{-}kCA(m)] = \mathcal{L}[XSV4\text{-}TM(\log m)]$ *for any* $X \in \{D, N\}$.

(**Proof**) (1) : Let M be an $XSV4\text{-}kCA(L(m))$. The set $T(M)$ is also accepted by the $XSV4\text{-}TM(\log L(m))$ which divides the storage tape into k tracks and makes each track play a role of the corresponding counter of M.

(2) : From (1), $\bigcup_{1\le k<\infty} \mathcal{L}\ [XSV4\text{-}kCA(m)] \subseteq \mathcal{L}[XSV4\text{-}TM(\log m)]$. It is well known that any $\log m$ tape-bounded one-dimensional off-line Turing machine can be simulated by a one-dimensional two-way multihead finite automaton [4]. By using the same argument as in the proof of this fact, we can easily show that any $XSV4\text{-}TM(\log m)$ can be simulated by an $XSV4\text{-}kCA(m)$ for some $k \ge 1$. Thus $\mathcal{L}[XSV4\text{-}TM(\log m)] \subseteq \bigcup_{1\le k<\infty} \mathcal{L}[XSV4\text{-}kCA(m)]$. □

2 Seven-Way versus Eight-Way

In this section, we investigate the difference between the accepting powers of counter-bounded eight-way and seven-way four-dimensional multicounter automata.

We need the following two lemmas.

[Lemma 2.1] *Let* $T_1 = \{x \in \{0,1\}^{(4)} \mid \exists m \ge 2\ [\ l_1(x) = l_2(x) = l_3(x) = l_4(x) = m]\ \&\ Cube_x(1) = Cube_x(2)\}$, *and let* $L_1(m) : \boldsymbol{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty} [(\log L_1(m))/m^3] = 0$. *Then,*

(1) $T_1 \in \mathcal{L}[D4\text{-}FA]$, *and*

(2) $T_1 \notin \bigcup_{1\le k<\infty} \mathcal{L}[NSV4\text{-}kCA(L_1(m))]$.

(**proof**) The proof of (1) is omitted here, since it is obvious. We now prove (2). Suppose that there is an $NSV4\text{-}TM(L_1'(m))$ M accepting T_1, where $L_1'(m) : \mathbf{N} \mapsto \mathbf{N}$ is a function such that $\lim_{m\to\infty} [L_1'(m)/m^3] = 0$. For each $m \ge 3$, let

$$V(m) = \{x \in T_1 \mid l_1(x) = l_2(x) = l_3(x) = l_4(x) = m \ \&\ x[(1,1,1,3),(m,m,m,m)] \in \{0\}^{(4)}\}.$$

Clearly, each tape in $V(m)$ is accepted by M. For any (seven-way) four-dimensional Turing machine M, we define the *configuration* of M to be a combination of the (1) state of the finite control, (2) position of the input head within the input tape, (3) position of the storage-tape head within the nonblank portion of the storage tape, and (4) contents of the storage tape. For each $x \in V(m)$, let $conf(x)$ be the set of configurations of M just after the point, in the accepting computations on x, where the input head left the first cube of x. Then the following proposition must hold.

[Proposition 2.1] *For any two different tapes* $x, y \in V(m)$,

$$conf(x) \cap conf(y) = \phi \ (\textit{empty set}).$$

(**Proof**) Suppose that $conf(x) \cap conf(y) \ne \phi$ and $\sigma \in conf(x) \cap conf(y)$. It is obvious that if, starting with this configuration σ, the input head proceeds to read the $[(1,1,1,2),(m,m,m,m)]$-segment of x, then M could enter an accepting state. Therefore, by assumption, it follows that the tape $z[\ l_1(z) = l_2(z) = l_3(z) = l_4(z) = m]$ satisfying the following two conditions must be also accepted by M: (i) $z[(1,1,1,1),(m,m,m,1)] = y[(1,1,1,1),(m,m,m,1)]$; (ii) $z[(1,1,1,2),(m,m,m,m)] = x[(1,1,1,2),(m,m,m,m)]$. This contradicts the fact that z is not in T_1. □

(**Proof of Lemma 2.1** (*continued*)) Clearly, $|V(m)| = 2^{m^3}$, where for any set S, $|S|$ denotes the number of elements of S. Let $c(m)$ be the number of possible configurations of M just after the input head left the first cube of tapes in $V(m)$. Then

$$c(m) \le s(m+2)^3 L_1'(m) t^{L_1'(m)}.$$

(The factor s is the number of possible states of finite control, $(m+2)^3$ is the number of possible positions of the input head, $L_1'(m)$ is the number of possible positions of the storage-tape head, t is the number of storage-tape symbols, and $t^{L_1'(m)}$ is the number of possible contents of the nonblank portion of storage tape.) Since $\lim_{m\to\infty} [L_1'(m)/m^3] = 0$, we have

$$|V(m)| > c(m) \text{ for large } m.$$

Therefore, it follows that for large m there must be different tapes $x, y \in V(m)$ such that $conf(x) \cap conf(y) \ne \phi$. This contradicts Proposition 2.1, and thus T_1 is not in $\mathcal{L}[NSV4\text{-}TM(L_1'(m))]$, where $L_1'(m) : \mathbf{N} \mapsto \mathbf{N}$ is a function such that $\lim_{m\to\infty} [L_1'(m)/m^3] = 0$. From this result and from the condition that $\lim_{m\to\infty} [(\log L_1(m))/m^3] = 0$, it follows that T_1 is not in $\mathcal{L}[NSV4\text{-}TM(\log L_1(m))]$. Part (2) of the lemma follows from this fact and Theorem 1.1 (1). □

[Lemma 2.2] *Let* $T_2 = \{x \in \{0,1\}^{(4)} \mid \exists m \ge 1\ [\ l_1(x) = l_2(x) = l_3(x) = l_4(x) = 2m\ \ \&\ \ x[(1,1,1,1),(2m,2m,2m,m)] = x[(1,1,1,m+1),(2m,2m,2m,2m)]$ (*that is, the top and bottom halves of x are identical*)$]\}$, *and let* $L_2(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L_2(m))/m^4] = 0$. *Then,*

(1) $T_2 \in \mathcal{L}[D4\text{-}1CA(m)]$, *and*

(2) $T_2 \notin \bigcup_{1\le k<\infty} \mathcal{L}[NSV4\text{-}kCA(L_2(m))]$.

(Proof) The proof of Part (1) is omitted here, since it is obvious. Part (2) is given by using the same technique as in the proof of Lemma 2.1 (2). □

From Lemmas 2.1 and 2.2, we can get the following theorem.

[Theorem 2.1] (1) *Let* $L(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L(m))/m^3] = 0$. *Then,* $\mathcal{L}[D4\text{-}FA] - \bigcup_{1\le k<\infty} \mathcal{L}[NSV4\text{-}kCA(L(m))] \neq \phi$. (2) *Let* $L'(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L'(m))/m^4] = 0$. *then,* $\mathcal{L}[D4\text{-}1CA(m)] - \bigcup_{1\le k<\infty} \mathcal{L}[NSV4\text{-}kCA(L'(m))] \neq \phi$.

3 Nondeterminism versus Determinism

In this section, we investigate the difference between the accepting powers of counter-bounded deterministic and nondeterministic seven-way four-dimensional multicounter automata.

We need the following two lemmas. The proof of the following lemmas is omitted here since it is similar to that of Lemma 2.1.

[Lemma 3.1] *Let* $T_3 = \{x \in \{0,1\}^{(4)} \mid \exists m \ge 2\ [\ l_1(x) = l_2(x) = l_3(x) = l_4(x) = m]\ \&\ Cube_x(1) \neq Cube_x(2)\}$, *and* $L_1(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L_1(m))/m^3] = 0$. *Then,*

(1) $T_3 \in \mathcal{L}[NSV4\text{-}FA]$, *and*

(2) $T_3 \notin \bigcup_{1\le k<\infty} \mathcal{L}[DSV4\text{-}kCA(L_1(m))]$.

[Lemma 3.2] *Let* $T_4 = \{x \in \{0,1\}^{(4)} \mid \exists m \ge 2\ [\ l_1(x) = l_2(x) = l_3(x) = l_4(x) = 2m\ \&\ x[(1,1,1,1),(2m,2m,2m,m)] \neq x[(1,1,1,m+1),(2m,2m,2m,2m)]]\}$, *and let* $L_2(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L_2(m))/m^4] = 0$. *Then,*

(1) $T_4 \in \mathcal{L}[NSV4\text{-}1CA(m)]$, *and*

(2) $T_4 \notin \bigcup_{1\le k<\infty} \mathcal{L}[DSV4\text{-}kCA(L_2(m))]$.

From Lemmas 3.1 and 3.2, we can get the following theorem.

[Theorem 3.1] (1) *Let* $L(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L(m))/m^3] = 0$. *Then,* $\mathcal{L}[NSV4\text{-}FA] - \bigcup_{1\le k<\infty} \mathcal{L}[DSV4\text{-}kCA(L(m))] \neq \phi$. (2) *Let* $L'(m) : \mathbf{N} \mapsto \mathbf{N}$ *be a function such that* $\lim_{m\to\infty}[(\log L'(m))/m^4] = 0$. *Then,* $\mathcal{L}[NSV4\text{-}1CA(m)] - \bigcup_{1\le k<\infty} \mathcal{L}[DSV4\text{-}kCA(L'(m))] \neq \phi$.

4 Conclusion

In this paper, we introduced four-dimensional multicounter automata, and we investigated the accepting powers of counter-bounded seven-way and eight-way four-dimensional multicounter automata. Then, we investigated a relationship between determinism and nondeterminism. In these subjects, we stated only for four-dimensional input tape which each sidelength is equivalent.

It will be also interesting to investigate the accepting powers of 'alternating' four-dimensional multicounter automata (see [2] for the concept of 'alternation').

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