Solving Constrained Motion Problems Using the GI Method[1]

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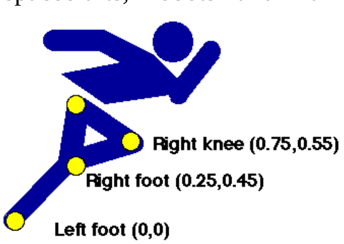
Abstract Derivation of equations of motion is the central part of analytical dynamics, which is important in the design of machines and prosthetic devices and in the motion control of spacecraft, robotic devices, and human bodies. This paper summarizes some recent developments of a new method for deriving equations of motion that was originally invented by Kalaba and Udwadia. Through simple examples, we demonstrate the simplicity of this method, its easy numerical implementation in modern computing environment, and its advantage of handling modification of constraints.

Key Words Analytical dynamics, nonideal nonholonomic constraints, generalized inverses of matrices, equation of motion

Introduction

Multibody dynamic mechanical systems occur in many classical and modern fields in science and engineering, such as in the design of vehicles, machines, and prosthetic devices, in the motion control of spacecrafts, robots and human bodies, and in the dynamic machinery control that can be integrated into active structural control against earthquake risk. For example, studies of human motion often begin with a simplified mechanical system of point masses and rigid bodies with constraints on motion. Flexible building structures (for protection against earthquakes) can be represented by a lumped system of point masses and springs. All efforts in modeling, simulating and controlling such dynamic systems start with the derivation of equations of motion, which is the central part of analytical dynamics.



As the classical field of analytical dynamics still serves as the theoretical foundation for many problems in modern science and technology, the complexity and large scale of most challenging and exciting new problems usually demand interdisciplinary collaboration and the aid of modern computers. The advanced mathematical concepts in classical Lagrangian mechanics are often difficult to understand outside of classical mechanics. In addition, most classical methods were developed before the computer era. Their formulation and derivation do not allow us to make the best use of the modern computing environment. Lastly, much of the classical analysis is based on the principle of virtual work, an assumption that Lagrange made to avoid thermodynamics concerns. Thus, constraints that involve forces that *do* work, such as friction, have been ruled out of classical methods. This limitation makes modeling difficult in situations where friction is not insignificant. For instance, in studies of the motion of sports-injured and the elderly, friction is often the driving force that causes great pain and limits the motion and cannot therefore be neglected. Therefore, new methods, together with extension and generalization of classical methods, are required for theoretical and practical reasons.

Recent development [1,2] on equations of motion for constrained mechanical systems opens possibilities for addressing the above mentioned limitations of classical methods. This new method exploits the advantages of the modern computing environment. It begins the analysis directly from the constraints of motion imposed on the systems, and arrives at an explicit set of equations of motion by using the chain rule of differentiation and the concept of generalized inverses (GI) of matrices. No generalized coordinate systems or any physical assumptions are required in the derivation. This new method handles *nonholonomic*[2] constraints with the same ease as holonomic constraints. More importantly, it takes into account the *nonideal*[3] constraints in a systematic and convenient manner. For brevity, the new method for deriving equations of motion will henceforth be referred to as the "GI method".

Recent Development of the GI Method

The original GI method [1] is equivalent to the classical methods such as Lagrange equation, Gibbs-Appel Equation, Hamilton Equation, and Gauss's

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[1] This special presentation is dedicated to Prof. Robert Kalaba (September 1926 - September 2004) by his last student Dr. Yueyue Fan. The work is based on their most recent collaboration on analytical dynamics.

[2] Nonholonomic constraints depend on time, displacement, and velocity, and are nonintegrable.

[3] Nonideal constraints involve forces that *do* work on the system in a virtual displacement.

principle of least constraint. However, the GI method has the advantages of providing an *explicit* equation of motion, easily handling nonholonomic constraints, and requiring no extra effort in treating dependent but consistent constraint equations. Later, the GI formula was further extended to systems including *non-ideal* constraints [3]. Further studies on its relation to other classical principles and its potential contribution to theory toward general dynamic and underdetermined systems are still ongoing [4].

Suppose that a mechanical system that contains p point masses is subjected to m holonomic or nonholonomic equality constraints of the form

$$f_i(x, \dot{x}, t) = 0, \; i = 1, 2, \ldots, m, \qquad (0)$$

where x is the displacement vector of the system of dimension $3p = n$. We also introduce the mass matrix M, which is of dimension $3p$ by $3p$, is a diagonal matrix, is positive definite, and has the masses $m_1, m_2, \ldots, m_p$ down the main diagonal in groups of three, with zeros elsewhere. As usual, $\dot{x}$ is the time derivative of x. Use of the chain rule of differentiation leads to a set of m equations that are *linear* in $\ddot{x}$, of the form

$$A\ddot{x} = b, \qquad (1)$$

where A is an m by $n = 3p$ matrix function of x, $\dot{x}$, and t, and b is an m by 1 column vector that may depend upon x, $\dot{x}$, and t. Given the initial conditions on x and $\dot{x}$, Eq. (1) is equivalent to Eq. (0).

If only ideal constraint forces are considered, it has been shown that the actual system acceleration vector is given by the explicit formula

$$\ddot{x} = a + M^{-1/2}(AM^{-1/2})^{+}(b - Aa), \qquad (2)$$

where $(AM^{-1/2})^{+}$ denotes the usual pseudoinverse of the matrix $AM^{-1/2}$. Vector a is the free motion acceleration if there were no constraint. Refer to reference [1] for the details.

Later, the GI formula was further extended to systems including *non-ideal* constraints [3]. The general equation of motion is

$$M\ddot{x} = F^{N} + F^{L} + F^{C}, \qquad (3)$$

where

$$F^{N} = Ma, \qquad (4)$$

$$F^{L} = M^{1/2}(AM^{-1/2})^{+}(b - Aa), \qquad (5)$$

$$F^{C} = M^{1/2}[I - (AM^{-1/2})^{+}AM^{-1/2}]M^{-1/2}c, \qquad (6)$$

and a is the free motion acceleration vector, and c is an arbitrary vector, both being of dimension 3n by 1. The notation recalls the names of Newton, Lagrange, and Coulomb. It has been shown in reference [3] that F^N is the newtonian impressed force vector, that F^L is a constraint force that does no work on the system in a virtual displacement v, and that F^C is a constraint force that *does* work on the system in a virtual displacement v. The type of force represented by F^C is called non-ideal constraint force, which includes sliding friction.

Eq. (3) is the most general possible equation of motion that is compatible with the constraint condition $A\ddot{x} = b$, assuming, of course, that the matrix M is nonsingular. Only two essential mathematical ideas are essential in the derivation of Eq. (3): the chain rule of differentiation and generalized inverses of matrices. Modern computing environments, such as Matlab, have built-in commands for calculating the generalized inverse of a matrix, so it makes the approach highly suitable for numerical studies. On the physical side the notions of mass, distance and time occur. There is no mention of kinetic energy, potential energy, moments, etc.. In the applications to specific systems, of course, the customary centripetal and Coriolis forces, moments, and so on do appear. These notions emerge naturally from the terms in the right side of Eq. (3), but no prior exposure to them is needed.

In classical analytical mechanics, it is assumed that the constraint force does no work in a virtual displacement. This means that the fundamental assumption of classical analytical mechanics is that $F^{C} = 0$, so that the equations of motion, Eq (3) reduce to Eq. (2). More generally, though, as in situations in which sliding friction is significant, we shall have $F^{C} \neq 0$, in which case the more general equation of motion, Eq. (3), will apply.

A Simple Example

A simple example is used to illustrate how the GI formula works. Consider a double pendulum system subjected to ideal constraints. The rectangular coordinates $(x_1\ y_1)$, $(x_2\ y_2)$ are as shown in Figure 1. The two constraints on the system are

$$x_1^2 + y_1^2 = l_1^2, \qquad (7)$$

and

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2^2. \qquad (8)$$

Eqs. (7) and (8) on two differentiations give

$$A\ddot{x} = b, \qquad (9)$$

where

$$A = \begin{bmatrix} x_1 & y_1 & 0 & 0 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 \end{bmatrix}, \qquad (10)$$

and

$$b = \begin{bmatrix} -(\dot{x}_1^2 + \dot{y}_1^2) \\ -[(\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2] \end{bmatrix}. \quad (11)$$

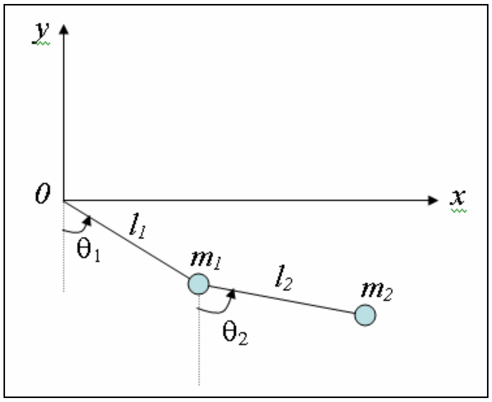

Figure 1: A Double Pendulum

Under ideal constraints, Eq. (2) is the equation of motion for this double pendulum system, with A and b defined by Eqs. (10) and (11). The mass matrix in this two-dimensional problem is

$$M = \begin{bmatrix} m_1 & 0 & 0 & 0 \\ 0 & m_1 & 0 & 0 \\ 0 & 0 & m_2 & 0 \\ 0 & 0 & 0 & m_2 \end{bmatrix}.$$

Let the initial position be $x_1 = l_1$, $y_1 = 0$, $x_2 = l_1 + l_2$, and $y_2 = 0$. Let the initial velocity of both particles be zero. The position and the velocity of the two particles at any time t can be obtained by integrating Eq. (2) with the given initial conditions.

Next, we will show how this double pendulum problem would have been solved in the classical Lagrangian mechanics. Lagrange considered mechanical systems as being characterized by potential energy, kinetic energy and the constraint function, with an emphasis on the use of generalized coordinates to describe the current configuration.

Let us use the two generalized coordinates θ_1 and θ_2 as shown in Figure 1. The virtual work done by the force of gravity is $m_1 g\delta y_1 + m_2 g\delta y_2$. But

$$y_1 = L_1 \cos\theta_1, \quad (12)$$

so that

$$\delta y_1 = L_1 \sin\theta_1 \delta\theta_1, \quad (13)$$

and

$$y_2 = L_1 \cos\theta_1 + L_2 \cos\theta_2, \quad (14)$$

so that

$$\delta y_2 = -(L_1 \sin\theta_1 \delta\theta_1 + L_2 \sin\theta_2 \delta\theta_2). \quad (15)$$

Using these expressions in the expression for virtual work, we get

$$Q_1\delta\theta_1 + Q_2\delta\theta_2 = -[m_1 g L_1 \sin\theta_1 \delta\theta_1 + m_2 g(L_1 \sin\theta_1 \delta\theta_1 + L_2 \sin\theta_2 \delta\theta_2)] = -(m_1 + m_2) g L_1 \sin\theta_1 \delta\theta_1 - m_2 g L_2 \sin\theta_2 \delta\theta_2. \quad (16)$$

Therefore, the generalized forces are

$$Q_1 = -(m_1 + m_2) g L_1 \sin\theta_1 \delta\theta_1, \quad (17)$$

and

$$Q_2 = -m_2 g L_2 \sin\theta_2 \delta\theta_2. \quad (18)$$

The kinetic energy can be written as

$$T = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\theta}_1^2 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + \frac{1}{2} m_2 L_2^2 \dot{\theta}_2^2. \quad (19)$$

The first Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} = Q_1 \quad (20)$$

yields

$$\frac{d}{dt}[(m_1 + m_2) L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)] + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) = -(m_1 + m_2) g L_1 \sin\theta_1, \quad (21)$$

and the second equation

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) - \frac{\partial T}{\partial \theta_2} = Q_2 \quad (22)$$

becomes

$$\frac{d}{dt}[m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)] - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) = -m_2 g L_2 \sin\theta_2. \quad (23)$$

Differentiating the left hand members of Eqs. (21, 23) with respect to time, we obtain the equations of motion of the system,

$$\begin{bmatrix} (m_1 + m_2) L_1^2 & m_2 L_1 L_2 \cos(\theta_1 - \theta_2) \\ m_2 L_1 L_2 \cos(\theta_1 - \theta_2) & m_2 L_2^2 \end{bmatrix} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{pmatrix} = \begin{pmatrix} -m_2 L_1 L_2 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g L_1 \sin\theta_1 \\ m_2 L_1 L_2 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - m_2 g L_2 \sin\theta_2 \end{pmatrix}. \quad (24)$$

By integrating Eq. (24) with the initial conditions $\theta_1(t_0) = 0$ and $\theta_2(t_0) = 0$, we obtain the results of the position of the two particles in the generalized coordinate system. These results, after being converted to the rectangular coordinate system, are compatible with the results from the GI formula.

In an ideal situation, the GI formula and the classical methods are equivalent. However, the derivation of the equations of motion using the classical method is quite complicated even for this simple example.

Handling Modification of Constraints

Most engineering design of mechanical systems involves modification of constraints on the base model. Handling such modification using the classical Lagrange mechanics requires change of the generalized coordinate system, and thus requires change of the generalized forces and the kinetic energy. Each time a constraint is modified, we will have to solve a completely new problem starting from the beginning. However, modification of constraints can be easily handled by the GI formula. Changing constraints of the system only changes matrix A and vector b in the equation of motion, Eq. (4). The rest of the procedure and the data required all remain the same. We will demonstrate this easy implementation using two examples below.

First, let us remove the first constraint in the double pendulum problem. Because only the second constraint is imposed on the system, as given in Eq. (8), only the second row of matrix A and vector b remains. Thus, we have

$$A = \left[x_1 - x_2 \quad y_1 - y_2 \quad x_2 - x_1 \quad y_2 - y_1 \right], \tag{25a}$$

and

$$b = -[(\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2]. \tag{25b}$$

Given initial position and velocity of the two particles, the problem can be solved by integrating the equation of motion with the new A and b.

Next, let us add an extra constraint to the original double pendulum problem

$$y_2 = -d \tag{26}$$

to keep the second particle moving along a horizontal line. Differentiate both sides of Eq. (26) twice, we have

$$\ddot{y}_2 = 0, \tag{27}$$

which adds a third row to the original constraints $A\ddot{x} = b$. The new A and b become

$$A = \begin{bmatrix} x_1 & y_1 & 0 & 0 \\ x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{28a}$$

and

$$b = \begin{bmatrix} -(\dot{x}_1^2 + \dot{y}_1^2) \\ -[(\dot{x}_2 - \dot{x}_1)^2 + (\dot{y}_2 - \dot{y}_1)^2] \\ 0 \end{bmatrix}. \tag{28b}$$

These are the only changes needed for solving the modified problem. The rest of the procedure remains the same. However, this one-degree-of-freedom problem cannot be easily handled if we were to use the classical methods. To express the kinetic energy and potential energy of the system in terms of single variable can be quite messy.

Conclusions and Discussion

In this paper, we have shown that the GI formula is suitable for the modern computing environment, and has potential to facilitate the analysis and control of large and complex mechanical systems. The only inputs required are the equations of the constraints and the initial conditions on the system. The rest of the procedure, such as differentiating the given functions, computing the generalized inverses of matrices, and integrating systems of differential equations, can be automated for execution by a computer. In order to fully utilize the advantage of the GI formula, a necessary step for future research is to automate the entire analysis of constrained mechanical systems. Full development of the new theory will require long-term cross-disciplinary collaboration from many scholars in mechanics, computational mathematics, and system optimization and control. We hope this paper will serve as an introduction of this GI method to the community of biomechanics and robotics design and will inspire further interest in applying and extending this new method.

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