Analysis and Control of a Novel 4D Chaotic System

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Abstract

In this paper, a novel four-dimensional (4D) autonomous chaotic system is presented. For chaos control of the 4D system, a linear feedback controller only with one variable is designed via matching the variable coefficients of the Lyapunov function, so that the system is no longer chaotic or periodic but globally asymptotically converges to the equilibrium point at the origin. The numerical simulation results are given to illustrate the feasibility and effectiveness of the method.

Keywords: the novel 4D chaotic system; chaos control; Lyapunov function with variable coefficients; global asymptotic stability

1. Introduction

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection,[1] which is a three-dimensional(3D) autonomous system and well known as the Lorenz system. Since then, many 3D chaotic systems have been proposed, such as the unified system[2], the Qi system[3] and so on. With the wide application of chaotic characteristics in secure communication and other fields, the complexity of chaotic systems is increasing. Four and higher dimensional chaotic systems have been investigated. [4-6] In this paper, a novel 4D chaotic system is presented.

Chaos control is one of the major subjects in chaos study, and the feedback control is a common method in chaos control. [7-10] In this paper, the aim of chaos control is to design a linear feedback controller and make the novel 4D chaotic system no longer chaotic or periodic but globally asymptotically stable at the origin.

The rest of this paper is organized as follows. In Section 2, the model of the novel 4D chaotic system and its simulation phase portraits are given. In Section 3, a Lyapunov function with variable coefficients is selected to design a linear feedback controller only with one feedback variable and prove the global asymptotic stability of the controlled 4D system. Some simulation results are given to demonstrate the validity of the linear feedback controller. The conclusions are drawn in Section 4.

2. The Novel 4D Chaotic System

The dynamic equations of the novel 4D chaotic system are formulated as

$$\begin{aligned} \dot{x} &= a\left(y - x\right), \\ \dot{y} &= c\left(x + y\right) + z - xw, \\ \dot{z} &= mx - y - hz, \\ \dot{w} &= xy - bw, \end{aligned} \tag{1}$$

where $x, y, z, w \in \mathbb{R}$ are state variables, and a = 25, b = 3, c = 18, m = 19 and h = 14.

Let the initial values of the 4D system (1) be (x_0, y_0, z_0, w_0) = (1, 1, 1, 1), then the Lyapunov exponents respectively are $\lambda_1 = 2.6686 > 0$,

$\lambda_2 = 0.0003 \approx 0$, $\lambda_3 = -11.7885 < 0$ and $\lambda_4 = -14.8804 < 0$. It indicates that the system (1) is chaotic. The phase portraits of the 4D system (1) are shown in Fig. 1(a1)-(a3).

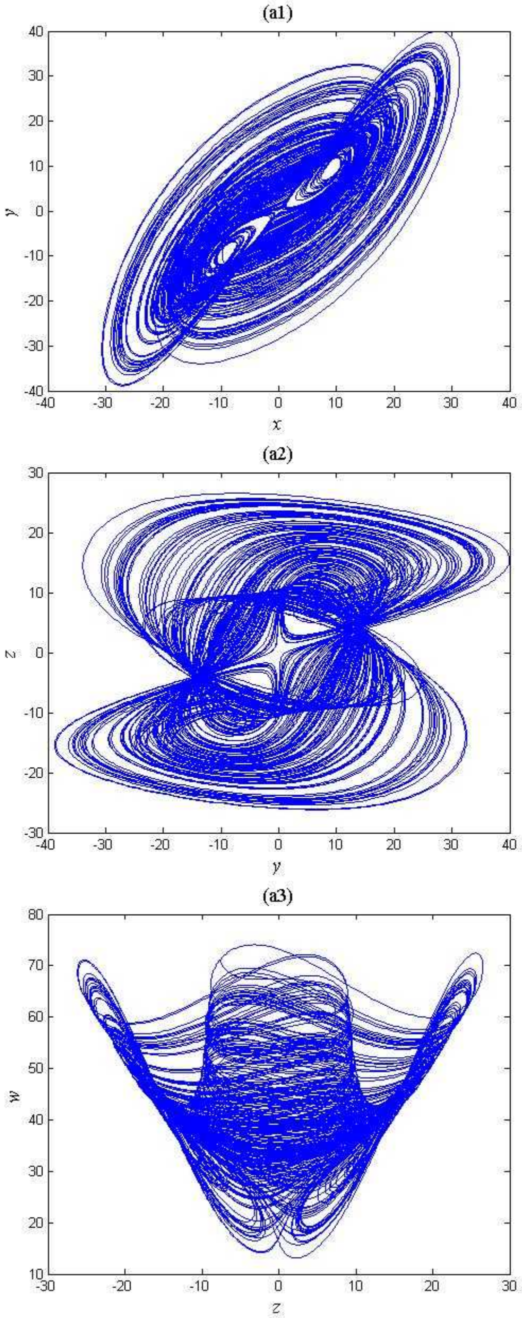

Fig. 1. Phase portraits of the novel 4D chaotic system: (a1) *x-y* (a2) *y-z* (a3) *z-w*

3. Chaos Control of the 4D Chaotic System

3.1. *Formulation of the controlled system*

Let $\tilde{\boldsymbol{X}} = \boldsymbol{X} - \mathbf{O} = \begin{bmatrix} \tilde{x} & \tilde{y} & \tilde{z} & \tilde{w} \end{bmatrix}^{\mathrm{T}}$ be the controlled state vector, where $\boldsymbol{X} = \begin{bmatrix} x & y & z & w \end{bmatrix}^{\mathrm{T}}$ is the state vector of the system (1), and $\mathbf{O} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$ is the origin. Then, the controlled system can be represented as

$$\begin{aligned} \dot{\tilde{x}} &= a(\tilde{y} - \tilde{x}) + u_{c1}, \\ \dot{\tilde{y}} &= c(\tilde{x} + \tilde{y}) + \tilde{z} - \tilde{x}\tilde{w} + u_{c2}, \\ \dot{\tilde{z}} &= m\tilde{x} - \tilde{y} - h\tilde{z} + u_{c3}, \\ \dot{\tilde{w}} &= \tilde{x}\tilde{y} - b\tilde{w} + u_{c4}, \end{aligned} \tag{2}$$

where

$$\begin{aligned} \boldsymbol{u}_{c} &= \begin{bmatrix} u_{c1} & u_{c2} & u_{c3} & u_{c4} \end{bmatrix}^{\mathrm{T}} \\ &= \begin{bmatrix} -k_1\tilde{x} & -k_2\tilde{y} & -k_3\tilde{z} & -k_4\tilde{w} \end{bmatrix}^{\mathrm{T}}, k_1, \cdots, k_4 \geq 0 \end{aligned} \tag{3}$$

is the linear feedback controller to be designed.

3.2. *Design of the controller*

Convert Eq. (2) to a state-space model which is expressed as

$$\dot{\tilde{\boldsymbol{X}}} = A\tilde{\boldsymbol{X}} + \boldsymbol{g}(\tilde{\boldsymbol{X}}), \tag{4}$$

where

$$A = \begin{bmatrix} -(k_1 + a) & a & 0 & 0 \\ c & -(k_2 - c) & 1 & 0 \\ m & -1 & -(k_3 + h) & 0 \\ 0 & 0 & 0 & -(k_4 + b) \end{bmatrix},$$

and $\boldsymbol{g}(\tilde{\boldsymbol{X}}) = \begin{bmatrix} 0 & -\tilde{x}\tilde{w} & 0 & \tilde{x}\tilde{y} \end{bmatrix}^{\mathrm{T}}$.

Take a positive definite function

$$\mathrm{V}(\tilde{\boldsymbol{X}}) = \frac{1}{2}\tilde{\boldsymbol{X}}^{\mathrm{T}} P \tilde{\boldsymbol{X}}$$

as a Lyapunov function candidate for the system (4), where

$$P = \begin{bmatrix} n_1 & 0 & 0 & 0 \\ 0 & n_2 & 0 & 0 \\ 0 & 0 & n_3 & 0 \\ 0 & 0 & 0 & n_4 \end{bmatrix}, n_1, n_2, n_3, n_4 > 0.$$

Then, the derivative $\dot{\mathrm{V}}(\tilde{\boldsymbol{X}})$ is given by

$$
\begin{aligned}
\dot{\mathrm{V}}\left(\tilde{\boldsymbol{X}}\right) &= \frac{1}{2}\tilde{\boldsymbol{X}}^{\mathrm{T}}\left(A^{\mathrm{T}}P+PA\right)\tilde{\boldsymbol{X}} \\
&+\frac{1}{2}\left[\boldsymbol{g}^{\mathrm{T}}\left(\tilde{\boldsymbol{X}}\right)P\tilde{\boldsymbol{X}}+\tilde{\boldsymbol{X}}^{\mathrm{T}}P\boldsymbol{g}\left(\tilde{\boldsymbol{X}}\right)\right] \\
&= \frac{1}{2}\tilde{\boldsymbol{X}}^{\mathrm{T}}Q\tilde{\boldsymbol{X}}+\left(n_4-n_2\right)\tilde{x}\tilde{y}\tilde{w}, \qquad (5)
\end{aligned}
$$

where

$$
Q=\begin{bmatrix} q_{11} & q_{12} & q_{13} & 0 \\ q_{21} & q_{22} & q_{23} & 0 \\ q_{31} & q_{32} & q_{33} & 0 \\ 0 & 0 & 0 & q_{44} \end{bmatrix},
$$

$$
\begin{aligned}
q_{11} &= -2n_1\left(k_1+a\right), \\
q_{22} &= -2n_2\left(k_2-c\right), \\
q_{33} &= -2n_3\left(k_3+h\right), \\
q_{44} &= -2n_4\left(k_4+b\right), \\
q_{12} &= q_{21} = an_1+cn_2, \\
q_{13} &= q_{31} = mn_3, \\
q_{23} &= q_{32} = n_2-n_3.
\end{aligned}
$$

Let $n_4 = n_2$, then Eq. (5) is simplified as

$$
\dot{\mathrm{V}}\left(\tilde{\boldsymbol{X}}\right)=\frac{1}{2}\tilde{\boldsymbol{X}}^{\mathrm{T}}Q\tilde{\boldsymbol{X}}.
$$

For the derivative $\dot{\mathrm{V}}\left(\tilde{\boldsymbol{X}}\right)$ to be negative definite, the leading principal minors of the matrix Q must satisfy

$$
\sigma_1 = -2n_1\left(k_1+a\right)<0, \qquad (6)
$$

$$
\begin{aligned}
\sigma_2 &= 4n_1n_2\left(k_1+a\right)\left(k_2-c\right) \\
&\quad -\left(an_1+cn_2\right)^2>0, \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_3 &= -8n_1n_2n_3\left(k_1+a\right)\left(k_2-c\right)\left(k_3+h\right) \\
&\quad +2mn_3\left(an_1+cn_2\right)\left(n_2-n_3\right) \\
&\quad +2n_2m^2n_3^2\left(k_2-c\right) \\
&\quad +2n_3\left(k_3+h\right)\left(an_1+cn_2\right)^2 \\
&\quad +2n_1\left(k_1+a\right)\left(n_2-n_3\right)^2<0, \qquad (8)
\end{aligned}
$$

$$
\sigma_4 = -2n_2\left(k_4+b\right)\sigma_3>0. \qquad (9)
$$

From Eq. (6), k_1 should satisfy $k_1 > -a$. Let $k_1 = 0$ and substitute it into Eq. (7), then k_2 should satisfy

$$
k_2 > \frac{\left(an_1+cn_2\right)^2}{4an_1n_2}+c. \qquad (10)
$$

Assume that the minimum number of the feedback variables might be equal to the number of the positive Lyapunov exponents.[11] The 4D chaotic system (1) only has one positive Lyapunov exponent and $k_2 \neq 0$ as shown in Eq. (10), so let $k_3 = 0$ and substitute $k_1 = k_3 = 0$ into Eq. (8). It is obtained that

$$
\begin{aligned}
\sigma_3 &= -8ahn_1n_2n_3\left(k_2-c\right) \\
&\quad +2mn_3\left(an_1+cn_2\right)\left(n_2-n_3\right) \\
&\quad +2n_2m^2n_3^2\left(k_2-c\right)+2hn_3\left(an_1+cn_2\right)^2 \\
&\quad +2an_1\left(n_2-n_3\right)^2<0. \qquad (11)
\end{aligned}
$$

Let $n_1 = n_2 = n_3$. From Eq. (11), k_2 should satisfy

$$
k_2 > \frac{h\left(a+c\right)^2}{4ah-m^2}+c, \qquad (12)
$$

so that Eq. (9) is satisfied as long as $k_4 > -b$. Hence, let $k_4 = 0$.

For $\dot{\mathrm{V}}\left(\tilde{\boldsymbol{X}}\right)$ to be negative definite, k_2 should satisfy both Eq. (10) and Eq.(12). It means that

$$
k_2 > \frac{h\left(a+c\right)^2}{4ah-m^2}+c > \frac{\left(a+c\right)^2}{4a}+c,
$$

when $n_1 = n_2 = n_3 = n_4$. Consequently, let $k_2 = 45$. As a result, because the derivative $\dot{\mathrm{V}}\left(\tilde{\boldsymbol{X}}\right)$ is negative definite and the Lyapunov function $\mathrm{V}\left(\tilde{\boldsymbol{X}}\right)$ is positive definite and radially unbounded, it can be confirmed that the controlled system (2) is globally asymptotically stable at the origin.

Substituting $k_2 = 45$ and $k_1 = k_3 = k_4 = 0$ into Eq. (3) yields

$$
\begin{aligned}
\boldsymbol{u}_{\mathrm{c}} &= \begin{bmatrix} u_{\mathrm{c}1} & u_{\mathrm{c}2} & u_{\mathrm{c}3} & u_{\mathrm{c}4} \end{bmatrix}^{\mathrm{T}} \\
&= \begin{bmatrix} 0 & -45\tilde{y} & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \qquad (13)
\end{aligned}
$$

3.3. *Numerical simulation*

Substitute Eq. (13) into Eq. (2) and let the initial values still be $\left(\tilde{x}_0,\tilde{y}_0,\tilde{z}_0,\tilde{w}_0\right)=\left(1,1,1,1\right)$, then the Lyapunov exponents of the controlled system (2) respectively are $\lambda_{\mathrm{c}1}=-3.0397$, $\lambda_{\mathrm{c}2}=-3.7495$, $\lambda_{\mathrm{c}3}=-15.3788$ and $\lambda_{\mathrm{c}4}=-46.8316$, which are all negative. It implies that the controlled system (2) is no longer chaotic or periodic but stable at the origin.

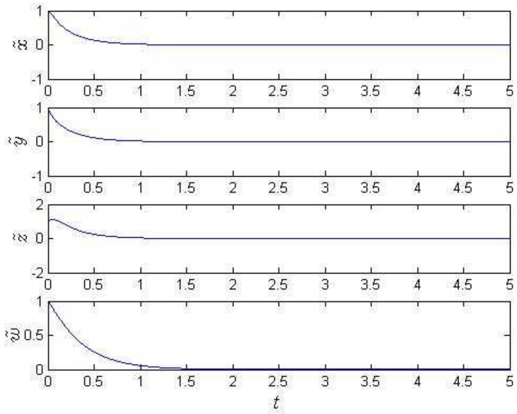

Fig. 2. Curves of the state variables of the controlled system

The curves of the state variables of the controlled system (2) are shown in Fig. 2. The horizontal axis t expresses the solution interval of differential equations, so t is a dimensionless quantity. From Fig. 2, it can be seen that the state variables $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ and $\tilde{w}$ converge to zero asymptotically and rapidly. It illustrates that the controller (13) is feasible and effective for chaos control of the novel 4D chaotic system (1).

4. Conclusions

In this paper, a novel 4D chaotic system is presented. The model of the 4D system and its chaotic attractor are complex and could be applied to secure communication. For chaos control of the 4D chaotic system, different groups of the coefficients of the Lyapunov function yield different linear feedback controllers. It makes the form and parameters of the linear feedback controller flexible to select. The linear feedback controller designed in this paper only has one feedback variable, so that it is easy to implement via circuit. Furthermore, the centers of the state variables of the 4D chaotic system have been translated to the origin before the controlled system is formulated. It means that this method can be used to make the system globally asymptotically converge to any point or even some specified states via center translation. Thus, this method could be applied to chaos synchronization of the novel 4D chaotic systems. It would be discussed in another paper. The study in this paper has some engineering significance.

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